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ON BIPARTITE SCORE SETS

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ABSTRACT

A necessary and sufficient condition for sets of non-negative integers $A=\{a\}$ and $B=\{b_1,b_2,\dots,b_n\}$, $0 \le b_1 < b_2 < \dots < b_n$ to be the score sets of a bipartite tournament is given.

A bipartite tournament is a complete asymmetric bipartite digraph. The number of edges oriented from a vertex is called a score. Two sequences $a_1 \le a_2 \le \dots \le a_k$ and $b_1 \le b_2 \le \dots \le$ $\dots \le b_\ell$, corresponding to the scores of the bipartite sets of a bipartite tournament, are called a score sequence. The sets $A=\{a_i \mid 1 \le i \le k\}$ and $B=\{b_i \mid 1 \le i \le \ell\}$ of elements of the score sequences, are called score sets.

Throughout the paper we shall denote by $A=\{a_1,a_2,\dots$ $\dots,a_m\}$ and $B=\{b_1,b_2,\dots,b_n\}$ sets of non-negative integers such that $a_1 < a_2 < \dots < a_m$ and $b_1 < b_2 < \dots < b_n$ where a_1 and b_1 are not both zero.

At the Fourth International Conference on Theory and Applications of Graphs in Kalamazoo 1981, K.B.Reid raised

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the problem of determining the score sets of bipartite tournaments. K.Wayland [4] found a necessary and sufficient condition for the existence of a bipartite tournament with bipartition (X,Y) and the score sets A and B, if $|X| > b_n$.

Since some bipartite tournaments exist only for $|X| = b_n$, it is very unlikely that a sensible necessary sufficient condition can be given for general case.

We present, using a constructive method, a necessary and sufficient condition for the existence of a bipartite tournament with the score sets $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

THEOREM. *The sets of non-negative integers* $A = \{a\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$ *are the score sets of some bipartite tournament if and only if one of the following conditions is satisfied:*

(a) $b_1+b_2+\ldots+b_n = (n-a-1)b_n$;

(b) $b_1+b_2+\ldots+b_n > (n-a-1)(b_n+1)$;

(c) $b_1+b_2+\ldots+b_n = (n-a-1)b_n+d$, $1 \le d \le n-a-1$

and there exist positive integers $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ *such that*

$$ab_n = \gamma_1(b_n-b_1) + \gamma_2(b_n-b_2) + \ldots + \gamma_{n-1}(b_n-b_{n-1}) .$$

P r o o f. Necessity. Firstly we prove the inequality

$$b_1+b_2+\ldots+b_{n-1} \ge (n-a-1)b_n . \tag{1}$$

Let T be a bipartite tournament whose bipartite sets X and Y have the score sequences

$$(\underbrace{a,a,\ldots,a}_{\alpha}) \text{ and } (\underbrace{b_1,\ldots,b_1}_{\beta_1},\underbrace{b_2,\ldots,b_2}_{\beta_2},\ldots,\underbrace{b_n,\ldots,b_n}_{\beta_n})$$

respectively, where $\alpha \ge 1$ and $\beta_i \ge 1$, $i=1,\ldots,n$. Denote by $y_{i1}, y_{i2}, \ldots, y_{i\beta_i}$, $i=1,\ldots,n$ vertices in Y having the score b_i and by $X_{i1}, X_{i2}, \ldots, X_{i\beta_i}$ their insets, i.e. $X_{ij} = \{x \mid x \to y_{ij}\}$. Since every vertex of X has a score a, the sets X_{ij}, $i=1,\ldots,n$, $j=1,\ldots,\beta_i$ are a covering of X such that every vertex of X is covered by precisely a insets. Thus,

$$|X_{i1}| = |X_{i2}| = \ldots = |X_{i\beta_i}| = |X| - b_i = \alpha - b_i, \quad i=1,\ldots,n$$

and

$$a\alpha = \sum_{i=1}^{n} \sum_{j=1}^{\beta_i} |X_{ij}| = \sum_{i=1}^{n} \beta_i(\alpha - b_i)$$

hold and we have

$$(2) \qquad \alpha = \frac{\beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_n b_n}{\beta_1 + \beta_2 + \ldots + \beta_n - a} .$$

As $\alpha \geq b_n$, we get from (2)

$$(3) \qquad \beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_{n-1} b_{n-1} \geq (\beta_1 + \beta_2 + \ldots + \beta_{n-1} - a) b_n .$$

From $\beta_i \geq 1$, $i=1,\ldots,n-1$ and $0 \leq b_1 < b_2 < \ldots < b_n$ (1) follows.

Now we prove that the equality

$$(4) \qquad b_1 + b_2 + \ldots + b_{n-1} = (n-a-1) b_n + d, \quad 1 \leq d \leq n-a-1$$

implies $ab_n = \gamma_1(b_n - b_1) + \gamma_2(b_n - b_2) + \ldots + \gamma_{n-1}(b_n - b_{n-1})$ for some positive integers γ_i, $i=1,\ldots,n-1$.

Let $y_1, y_2, \ldots, y_n$ be vertices in y with the scores $b_1, b_2, \ldots, b_n$ and $X_1, X_2, \ldots, X_n$ their insets. Denote by s_o the total number of vertices in all other insets and by s_i the cardinality of X_i, $i=1,\ldots,n$. Then the equalities

$$b_i = \alpha - s_i, \quad i=1,\ldots,n$$

$$s_1 + s_2 + \ldots + s_n + s_o = a\alpha$$

hold and imply

$$(5) \qquad b_i = (s_1 + \ldots + s_{i-1} + (1-a)s_i + s_{i+1} + \ldots + s_n + s_o)/a, \quad i=1,\ldots,n.$$

Substituting (5) in (4) we get

$$(n-a)s_n + s_o = d$$

and, since $s_n \geq 0$ and $1 \leq d \leq n-a-1$ (in particular $a < n$), we get

$$s_n = 0$$

i.e.

$$b_n = \alpha .$$

Hence (3) becomes an equality and hence

$$ab_n = \beta_1(b_n-b_1) + \beta_2(b_n-b_2) + \ldots + \beta_{n-1}(b_n-b_{n-1})$$

where $\beta_i \geq 1$, $i=1,\ldots,n-1$.

Setting $\gamma_i = \beta_i$, $i=1,\ldots,n$, we prove the statement.

Sufficiency. The structure of a bipartite tournament with the given bipartite sets X and Y is determined by the outsets of all the vertices of Y. We shall just construct these outsets.

According to the theorem we have to consider three cases.

(a) $b_1+b_2+\ldots+b_{n-1} = (n-a-1)b_n$.

Let T be the bipartite tournament with the bipartite sets $X = \{1,2,\ldots,b_n\}$ and $Y = \{y_1,y_2,\ldots,y_n\}$ and the dominance structure

$$y_i \to \{b_1+b_2+\ldots+b_{i-1}+1,\ldots,b_1+b_2+\ldots+b_{i-1}+b_i\},\ i=1,\ldots,n.$$

Summarizing is modulo b_n. Clearly the score of y_i, $i=1,\ldots,n$ is $s(y_i) = b_i$. From the construction and the fact that

$$\sum_{i=1}^{n} s(y_i) = \sum_{i=1}^{n} b_i = (n-a)b_n ,$$

it follows that every vertex of X is dominated by precisely $(n-a)$ vertices of Y. As $|Y| = n$, the scores of all the vertices of X are a, and T is required tournament.

(b) $b_1+b_2+\dots+b_{n-1} = (n-a-1)(b_n+1)+d, \quad d \geq 1$.

Set $X=\{1,2,\dots,b_n+1\}$ and $Y=\{y_1,y_2,\dots,y_{n-1},z_1,z_2,\dots,z_d\}$, and construct the bipartite tournament T with the bipartite sets X and Y as follows:

$$y_i \to \{b_1+b_2+\dots+b_{i-1}+1,\dots,b_1+b_2+\dots+b_{i-1}+b_i\}$$

for $i=1,\dots,n-1$ and

$$z_j \to \{b_1+\dots+b_{n-1}+(j-1)b_n+1,\dots,b_1+\dots+b_{n-1}+jb_n\}$$

for $j=1,\dots,d$. Summarizing is modulo b_n+1.

Now $s(y_i)=b_i$, $i=1,\dots,n-1$ and $s(z_j)=b_n$ $j=1,\dots,d$. The equality

$$\sum_{i=1}^{n-1} s(y_i) + \sum_{j=1}^{d} s(z_j) = \sum_{i=1}^{n-1} b_i + d\,b_n = (n-a-1+d)(b_n+1)$$

implies that every vertex x of X is dominated by exactly $n-a-1+d$ vertices of Y and, therefore, has the score

$$\begin{aligned} s(x) &= |Y| - (n-a-1+d) \\ &= (n-1+d) - (n-a-1+d) = a \end{aligned}$$

This proves the construction.

(c) $b_1+b_2+\dots+b_{n-1} = (n-a-1)b_n+d, \quad 1 \leq d \leq n-a-1$

and

$$ab_n = \gamma_1(b_n-b_1)+\gamma_2(b_n-b_2)+\dots+\gamma_{n-1}(b_n-b_{n-1}), \quad \gamma_i \geq 1 ,$$
$i=1,\dots,n-1$.

In this case let $X=\{1,2,\dots,b_n\}$,

$$Y=\{y_n,\dots,y_{1\gamma_1},y_{21},\dots,y_{2\gamma_2},\dots,y_{n-1,1},\dots,y_{n-1,\gamma_{n-1}},y_n\}$$

and

$$y_{ij} \to \{\gamma_1 b_1+\dots+\gamma_{i-1}b_{i-1}+(j-1)b_i+1,\dots,\gamma_1 b_1+\dots+\gamma_{i-1}b_{i-1}+jb_i\}$$

for $i=1,\dots,n-1$, $j=1,\dots,\gamma_i$

$$y_n \to \{\gamma_1 b_1+\dots+\gamma_{n-1}b_{n-1}+1,\dots,\gamma_1 b_1+\dots+\gamma_{n-1}b_{n-1}+b_n\}$$

Summarizing is modulo b_n. Similarly as, in (a) and (b), we obtain that $s(y_{ij}) = b_i$, $i=1,\dots,n-1$, $s(y_n) = b_n$ and $s(x) = a$ for every $x \in X$.

This proves the theorem.

CORROLARY. *(Wayland [4]). Any finite nonempty set of non-negative integers, except {0}, may be the union of the score sets of some bipartite tournament.*

P r o o f. Let $a_1, a_2, \dots, a_n$ be a set of nonnegative inregers such that $0 \leq a_1 < a_2 < \dots < a_n$. Set $A = a_n$, and $B = \{a_1, a_2, \dots, a_{n-1}\}$. Since $a_i \geq i-1$, $i=1,\dots,n$, particularly $a_n \geq n-1$, the following inequality

$$a_1+a_2+\dots+a_{n-2} > ((n-1)-a_n-1)(a_{n-1}+1)$$

holds.

According to the case (b), there exists a bipartite tournament with bipartite sets X and Y whose score sets are $\{a_n\}$ and $\{a_1, a_2, \dots, a_{n-1}\}$, respectively.

REFERENCES

[1] L.W.Beineke, J.W.Moon, *On Bipartite Tournaments and Scores, The Theory and Applications of Graphs, Fourth International Conference Western Michigan University, Kalamazoo.pp.55-71, John Wiley, 1981.*

[2] J.W.Moon, *On the score sequence of an n-partite tournament. Canadian Mathematical Bulletin, Vol.5, No. 1 Jan.1962,pp. 51-58.*

[3] C.Thomassen, private communication, 1984.

[4] K.Wayland, *Bipartite score sets, Canadian Mathematical Bulletin, Vol. 26, No. 3, 1983, pp. 273-279.*

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REZIME

O SKUPOVIMA SKOROVA BIPARTITNOG TURNIRA

U ovom radu daje se potreban i dovoljan uslov za skupove nenegativnih celih brojeva, $A = \{a\}$ i $B = \{b_1, b_2, \ldots, b_n\}$ $0 \leq b_1 < b_2 < \ldots < b_n$, da budu skupovi skorova nekog bipartitnog turnira.