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SOME COMBINATORIAL IDENTITIES INSPIRED BY THE LAW OF ENERGY

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ABSTRACT

The investigations in paper [1] and the law of energy resulted combinatorials identities (1), (4) and (5) the proofs of which were indispensable for the physical interpretations of qualitative and quantitative points of the above mentioned paper. In this paper the identities (1), (4) and (5) are proved.

THEOREM 1.

$$(1)\qquad \sum_{i=0}^{s} (-1)^{i} \binom{n}{s-i} \binom{p-s+i}{i} = \binom{n-p+s-1}{s} \quad \textit{for each}$$

$p,s,n \in N \cup \{0\}$ *and* $0 \le s \le p < n$.

P r o o f. Denote the functions on the left and right side of identity (1) as follows

$$F(p,s,n) = \sum_{i=0}^{s} (-1)^{i} \binom{n}{s-i} \binom{p-s+i}{i} \text{ and } G(p,s,n) = \binom{n-p+s-1}{s}.$$

The proofs are given by induction according to p. The theorem is obviously true for p=0 and p=1 and for each s and n, $0 \le s \le p < n$. Suppose that $F(p,s,n) = G(p,s,n)$ is true for some p and each s and n, $0 \le s \le p < n$. Now, we shall prove that $F(p+1,s,n) = G(p+1,s,n)$ for each s and n, $0 \le s \le p+1 < n$. First

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we shall prove that $F(p+1,s,n) = G(p+1,s,n)$ for each s and n, $0 \le s \le p < n$ and then for $s = p+1$. Let us prove that

(2) $F(p+1,s,n) = F(p,s,n) - F(p,s-1,n)$, for each $0 \le s \le p < n$.

Proof of (2)

$$F(p,s,n)-F(p,s-1,n) = \sum_{i=o}^{s} (-1)^i \binom{n}{s-i}\binom{p-s+i}{i} - \sum_{i=o}^{s-1} (-1)^i \binom{n}{s-1-i}\binom{p-s+1+i}{i} =$$

$$= \binom{n}{s}\binom{p-s}{0} + \sum_{i=1}^{s} (-1)^i \binom{n}{s-i}\binom{p-s+i}{i} - \sum_{i=0}^{s-1} (-1)^i \binom{n}{s-1-i}\binom{p-s+1+i}{i} =$$

$$= \binom{n}{s}\binom{p-s}{0} + \sum_{i=0}^{s-1} (-1)^{i+1} \binom{n}{s-1-i}\binom{p-s+i+1}{i+1} +$$

$$+ \sum_{i=0}^{s-1} (-1)^{i+1} \binom{n}{s-1-i}\binom{p-s+i+1}{i} = \binom{n}{s}\binom{p-s}{0} +$$

$$+ \sum_{i=0}^{s-1} (-1)^{i+1} \binom{n}{s-1-i}\binom{p+1-s+i+1}{i+1} = \binom{n}{s}\binom{p+1-s}{0} +$$

$$+ \sum_{i=1}^{s} (-1)^i \binom{n}{s-i}\binom{p+1-s+i}{i} =$$

$$= \sum_{i=0}^{s} (-1)^i \binom{n}{s-i}\binom{p+1-s+i}{i} = F(p+1,s,n) .$$

We also need to prove that

(3) $G(p+1,s,n) = G(p,s,n) - G(p,s-1,n)$, for each $0 \le s \le p < n$.

Proof of (3)

$$G(p,s,n) - G(p,s-1,n) = \binom{n-p+s-1}{s} - \binom{n-p+s-2}{s-1} =$$

$$= \binom{n-(p+1)+s-1}{s} = G(p+1,s,n) .$$

The equality $F(p+1,s,n) = G(p+1,s,n)$ for each $0 \le s \le p < n$, follows from (2) and (3), bearing in mind that $F(p,s,n) = G(p,s,n)$ and $F(p,s-1,n) = G(p,s-1,n)$ for each $0 \le s \le p < n$.

The equality $F(p+1,p+1,n) = G(p+1,p+1,n)$, for each $p+1 < n$ is the consequence of the following consideration:

$$\sum_{i=0}^{p+1}(-1)^i\binom{n}{p+1-i} = \sum_{i=0}^{p}(-1)^i\left(\binom{n-1}{p+1-i}+\binom{n-1}{p-i}\right) +$$

$$+ (-1)^{p+1}\binom{n}{0} = \binom{n-1}{p+1}+\binom{n-1}{p}-\binom{n-1}{p}-\binom{n-1}{p-1}+\binom{n-1}{p-1}+\ldots$$

$$+\ldots+(-1)^p\binom{n-1}{0}+(-1)^{p+1}\binom{n}{0}=\binom{n-1}{p+1}$$

THEOREM 2.

$$(4)\qquad \sum_{s=0}^{p}\binom{n}{p-s}(1-x)^s x^{p-s} = \sum_{s=0}^{p}\binom{n-p+s-1}{s}x^s ,$$

for each $p,s,n \in N \cup \{0\}$, $0 \leq s \leq p < n$ *and* $x \in R$.

P r o o f. Let us multiply (1) by x^s and perform the summation from $s=0$ to $s=p$. Hence

$$\sum_{s=0}^{p}\sum_{i=0}^{s}(-1)^i\binom{n}{s-i}\binom{p-s-i}{i}x^s = \sum_{s=0}^{p}\binom{n-p+s-1}{s}x^s .$$

By substituting $s=k+i$ we change the order of summation on the left side L where it is obvious that the s run from $s=k$ to $s=p$ and i from $i=0$ to $i=p-k$, for fixed k. Hence

$$L = \sum_{k=0}^{p}\sum_{i=0}^{p-k}(-1)^i\binom{n}{k}\binom{p-k}{i}x^{k+i} = \sum_{k=0}^{p}\binom{n}{k}x^k\sum_{i=0}^{p-k}(-1)^i\binom{p-k}{i}x^i =$$

$$= \sum_{k=0}^{p}\binom{n}{k}x^k(1-x)^{p-k}$$

further, taking $p-k=s$, we have:

$$L = \sum_{s=0}^{p}\binom{n}{p-s}x^{p-s}(1-x)^s$$

the theorem is proved.

THEOREM 3.

$$(5)\quad \sum_{j=0}^{r-1} \binom{r-1}{j} (1-x)^j x^{r-1-j} \sum_{k=0}^{j} \frac{y^k}{k!} = 1 + \sum_{j=0}^{r-2} \sum_{k=0}^{j} \binom{j}{k} (1-x)^{k+1} x^{j-k} \frac{y^{k+1}}{(k+1)!},$$

for each $x,y \in R$ *and* $r \in N\setminus\{1\}$.

P r o o f. If in (4) we put $p=r-k-2$ and $n=r-1$, then $j-k-1=s$ and $j-k=s$ change, introduced for the left and right side of the identity respectively, the following result is obtained:

$$(6)\quad \sum_{j=k+1}^{r-1} (1-x)^{j-k-1} x^{k-1-j} = \sum_{j=k}^{r-2} \binom{j}{k} x^{j-k}$$

because $p-s=r-k-2-s=r-1-k-1-s=r-1-j$, $\binom{n}{p-s}=\binom{r-1}{r-1-j}=$

$=\binom{r-1}{j}$, $\binom{n-p+s-1}{s}=\binom{r-1-r+k+2+s}{s}=\binom{k+s}{s}=\binom{j}{s}=\binom{j}{j-s}=\binom{j}{k}$.

The identity (6) is multiplied by

$$\frac{(1-x)^{k+1} y^{k+1}}{(k+1)!}$$

and the summation from k=0 to k=r-2 is performed, one is added to the left and right side, and the following equality is obtained:

$$1 + \sum_{k=0}^{r-2} \frac{y^{k+1}}{(k+1)!} \sum_{j=k+1}^{r-1} \binom{r-1}{j} (1-x)^j x^{r-1-j} = 1 + \sum_{k=0}^{r-2} \frac{y^{k+1}}{(k+1)!} \sum_{j=k}^{r-2} \binom{j}{k} x^{j-k} (1-x)^{k+1}.$$

Taking k-1 instead of k, we obtain:

$$(7)\quad \sum_{k=0}^{r-1}\sum_{j=k}^{r-1}\binom{r-1}{j}(1-x)^{j}x^{r-1-j}\frac{y^{k}}{k!} =$$

$$= 1 + \sum_{k=0}^{r-2}\sum_{j=k}^{r-2}\binom{j}{k}(1-x)^{k+1}x^{j-k}\frac{y^{k+1}}{(k+1)!},$$

because $\sum_{j=0}^{r-1}\binom{r-1}{j}(1-x)^{j}x^{r-1-j} = 1.$

Theorem 3 is obtained by changing the summation order on both sides of equality (7).

REFERENCES

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REZIME

NEKI KOMBINATORNI IDENTITETI INSPIRISANI ZAKONOM ENERGIJE

Rezultati rada [1] i zakon o održanju energije nameću kombinatorne identitete (1),(4) i (5) čiji dokazi su bili neophodni za fizičku interpretaciju kvalitativnih i kvantitativnih poenti rada [1], a koji su dati u ovom radu.