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A COUNTERFEIT COINS PROBLEM

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ABSTRACT

We consider the problem of ascertaining the minimum number of weighings which suffice to determine all counterfeit (heavier) coins in a set of n coins of the same appearance, given a balance scale and the information that there are exactly three heavier coins present. A procedure which is either optimal or suboptimal is constructed for infinitely many n's, i.e., for all $n = 3^k$ $(k=1,2,3,\dots)$.

1. INTRODUCTION

Consider the following problem. Let $X = \{c_1, c_2, \dots, c_n\}$ be a set of n coins indistinguishable except that exactly m $(m \le n)$ of them are slightly heavier than the rest (in the sense specified below). Given a balance scale, we want to find an optimal weighing procedure, i.e., a procedure which minimizes the maximum number of steps (weighings) which are required to identify all heavier coins. For some discussion of these matters in greater detail, see [1], [2], [3], [4] and [5].

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We suppose that all heavier coins are of equal weight, and so are all light coins. If λ is the weight of a light (good) coin, then the weight of a heavy (counterfeit) coin is less than $\frac{m+1}{m}\lambda$, so that the larger of the two numerically unequal subsets of X is always the heavier. This means that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of two subsets of X is heavier but not by how much.

Consider a pair (A,B) of numerically equal disjoint subsets of X. Step (A,B) will mean the balancing of A against B. The following outcomes are possible:

(a) The sets balance, symbolized by $A = B$,

(b) The sets do not balance, symbolized by $A \neq B$. We use the notation, if necessary, $A > B$, $A < B$, where $>$ and $<$ between two sets means "is heavier than" and "is lighter than" respectively.

Let $P_n^m(\ell)$ denote any procedure which enables us to identify all heavier coins, if there are exactly m of them in the set of n coins, ℓ being the maximum number of weighings to be required. $P_n^m(\leq \ell)$ will mean a procedure for which the maximum number of steps to be required is not greater than ℓ. A procedure $P_n^m(\ell)$ is said to be optimal if no one procedure $P_n^m(r)$ exists for some $r < \ell$. We write $\mu_m(n) = \ell$ if there is an optimal procedure $P_n^m(\ell)$. A procedure is said to be suboptimal if $\mu_m(n) = \ell-1$. It follows by information-theoretical reasonings that

$$\mu_m(n) \geq \lceil \log_3 \binom{n}{m} \rceil$$

where $\lceil x \rceil$ denotes the least integer $\geq x$. It is well known that $\mu_1(n) = \lceil \log_3 n \rceil$. In [6] it is proved that

$$\lceil \log_3 \binom{n}{2} \rceil \leq \mu_2(n) \leq 1 + \lceil \log_3 \binom{n}{2} \rceil$$

and a corresponding procedure is construced such that the lower bound is reached for an infinite set of n's.

In this paper, a procedure for three counterfeit coins problem is constructed, which is either optimal or suboptimal for infinitely many n's, i.e., for all $n = 3^k$ (k=1,2,3,...).

2. RESULTS

THEOREM. *If* $n = 3^k$ (k=1,2,3,...), *then*

$$\lceil \log_3 \binom{n}{3} \rceil \le \mu_3(n) \le 1 + \lceil \log_3 \binom{n}{3} \rceil .$$

P r o o f. It is easy to check that $3^{3k-2} < \binom{3^k}{3} < 3^{3k-1}$, i.e., $\lceil \log_3 \binom{3^k}{3} \rceil = 3k-1$, for $k \ge 2$. Now, the statement will be proved by the inductive construction of a procedure $P^3_{3^k}$ ($\le 3k$), for $k \ge 1$.

For $k = 1$, we have the trivial (empty) strategy $P^3_3(0)$ which satisfies the statement.

Suppose that a procedure $P^3_{3^k}$ ($\le 3k$) is constructed. Then, a procedure $P^3_{3^{k+1}}$ ($\le 3k+3$) can be constructed as follows.

Let $A = \{c_1, \ldots, c_{3^k}\}$, $B = \{c_{3^k+1}, \ldots, c_{2\cdot 3^k}\}$, $C = \{c_{2\cdot 3^k+1}, \ldots, c_{3^{k+1}}\}$.

Step 1. (A,B).
Step 2. (B,C).

It is sufficient to analyse four cases ((a)-(d) below); any other possible case is quite analogous to one of these four.

(a) $A = B$, $B = C$. It is clear that each of the sets A,B,C, contains exactly one heavier coin. We continue by successive applications of the procedure $P^1_{3^k}(k)$, three times, to the sets A,B and C independently. It follows that all heavier coins will be found after 3k+2 steps.

(b) $A = B$, $B < C$. Now, all heavier coins are in the set C.

We apply a procedure $P^3_{3^k}$ $(\leq 3k)$, which can be constructed by the induction hypothesis, to the set C. All heavier coins will be found after at most 3k+2 steps.

(c) $A < B$, $B < C$. We conclude that one heavier coin is in the set B and two of them are in the set C. They all can be found by applying two independent procedures, $P^1_{3^k}(k)$ and $P^2_{3^k}(2k)$, to the sets B and C respectively. The construction of a procedure $P^2_{3^k}(2k)$ is given in [6]. So, all heavier coins will be found after 3k+2 steps.

(d) $A < B$, $B > C$. Go to Step 3.

Step 3. (A,C).

There are three possible cases.

(da) $A = C$. We conclude that all heavier coins are in the set B, and continue by the application of a procedure $P^3_{3^k}(\leq 3k)$ to the set B. All heavier coins will be found after at most 3k+3 steps.

(db) $A < C$. Now, one heavier coin is in the set C and two of them are in the set B. We continue similarly as in the case (c), by applying two procedure, $P^1_{3^k}(k)$ and $P^2_{3^k}(2k)$, to the sets C and B respectively. All the heavier coins will be found after 3k+3 steps.

(dc) $A > C$. This case is quite similar to the case (db); now, one heavier coin is in the set A and two of them are in the set B, so, 3k+3 steps will suffice.

A procedure $P^3_{3^{k+1}}$ $(\leq 3k+3)$ is constructed and the theorem is proved.

REMARK. It is easy to see that, for $k \geq 3$, the constructed procedure is in fact a $P^3_{3^k}(3k)$ procedure; only for $k = 1$ and $k = 2$, it is a $P^3_{3^k}(< 3k)$ procedure. For $k = 1$, we have the trivial procedure $P^3_3(0)$; for $k = 2$, $P^2_{3^k}(2k)$ which is used in (db) and (dc) become $P_3^2(2)$ and can be replaced by a $P_3^2(1)$ procedure, while $P^3_{3^k}(\leq 3k)$ used in (da) become $P_3^3(0)$. So, for $k=2$, we obtain a procedure $P_9^3(5)$, which is optimal since the information-theoretical lower bound is reached.

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REZIME

JEDAN PROBLEM O NEISPRAVNIM NOVČIĆIMA

Posmatra se problem odredjivanja minimalnog broja merenja dovoljnih za identifikaciju svih neispravnih (težih) novčića u skupu od n novčića, uz pretpostavku da se u tom skupu nalaze tačno tri neispravna novčića. Jedna procedura koja je optimalna ili suboptimalna, konstruisana je za jedan beskonačan skup vrednosti parametra n.