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# PROBABILISTIC METRIC STRUCTURES: TOPOLOGICAL CLASSIFICATION

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**ABSTRACT**

A characterization of the topologies induced by each of several special types of pre-probabilistic metric structures is given. Regarding metrization, we propose an axiom ($IV_h$) which is easy to verify and at the same time avoids the notion of a t-norm.

## 1. INTRODUCTION

In numerous instances in which the theory of metric spaces is applied, the association of a single number as the distance between a pair of elements is rather an over-idealization. Probabilistic metric spaces render the concept of distance as a probabilistic rather than a determinate one. Two basic problems in the theory of probabilistic metric spaces have been dealt with rather unsatisfactorily. One of these is the manner in which topologies are associated with probabilistic metric structures which resulted in the study of several types of generalized topologies. The other is the approach to the metrization problem in probabilistic metric spaces. A sufficient condition for the metrization of a pro-

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babilistic metric space postulates the existence of a certain kind of t-norm. Axiom ($IV_h$) avoids this notion. Each probabilistic metric space naturally gives rise to neighbourhood-like filters, which, in general, are not neighbourhood filters for any topology. This led to the study of generalized topologies. We show that each probabilistic metric space gives rise to a topology in a very natural way and therefore it is not necessary to consider any generalized topologies.

The authors are not suggesting that the idea of a t-norm (or a triangle function) be abandoned. This idea, along with some form of the triangle inequality, leads to the intrinsic geometry of probabilistic metric spaces.

DEFINITION 1.1. *A function* $F: R \to [0,1]$ *is a distribution function if it is a non-decreasing, left continuous function with* $\inf F = 0$ *and* $\sup F = 1$.

DEFINITION 1.2. *Let* X *be a set and* $\mathcal{F}$ *be a function on* $X \times X$ *such that* $\mathcal{F}(x,y) = F_{xy}$ *is a distribution function. Consider the following conditions:*

I. $F_{xy}(0) = 0$ *for all* $x, y$ *in* X.

II. $F_{xy}(\varepsilon) = 1$ *for all* $\varepsilon > 0$ *iff* $x = y$.

III. $F_{xy} = F_{yx}$.

IV. *If* $F_{xy}(\varepsilon) = 1$ *and* $F_{yz}(\delta) = 1$, *then* $F_{xz}(\varepsilon+\delta) = 1$.

$IV_h$. *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $1-F_{xy}(\delta) < \delta$ *and* $1-F_{yz}(\delta) < \delta$ *then* $1-F_{xz}(\varepsilon) < \varepsilon$.

If $\mathcal{F}$ satisfies conditions I and II then it is called a pre-probabilistic metric structure (PPM-structure) on X and the pair $(X, \mathcal{F})$ is called a pre-probabilistic metric space (PPM-space). An $\mathcal{F}$ satisfying condition III is said to be symmetric. A symmetric PPM-structure $\mathcal{F}$ satisfying IV is a probabilistic metric structure (PM-structure) and the pair $(X, \mathcal{F})$ is a probabilistic metric space (PM-space). A symmetric PPM-structure

$F$ satisfying $IV_h$ is called an H-structure and the pair $(X,F)$ is called an H-space. The quantity $F_{xy}(\varepsilon)$ is to be interpreted as the probalility that the distance from x to y is less than $\varepsilon$.

DEFINITION 1.3. *A real-valued function* T *on the unit square* $I^2$ *is called a* t-*norm if it satisfies the following conditions:*

1. $0 \leq T(a,b) \leq 1$.
2. $T(a,b) = T(b,a)$.
3. $T(a,1) = a$.
4. $T(a,b) \leq T(c,d)$ *if* $a \leq c$ *and* $b \leq d$.
5. $T(T(a,b),c) = T(a,T(b,c))$.

DEFINITION 1.4. *A PM-space* $(X,F)$ *is called a Menger space if there exists a* t-*norm* T *such that*

$IV_m$: *for all* $x,y,z \in X$ *and for all* $r,s \geq 0$

$$F_{xz}(r+s) \geq T(F_{xy}(r),\ F_{yz}(s)).$$

## 2. TOPOLOGIES INDUCED BY PPM-STRUCTURES

Let $(X,F)$ be a PPM-space. For $\varepsilon,\lambda > 0$ and $x \in X$, define

$$V(\varepsilon,\lambda) = \{(y,z): 1-F_{yz}(\varepsilon) < \lambda\},$$

and

$$N_x(\varepsilon,\lambda) = \{y: (x,y) \in V(\varepsilon,\lambda)\}\ .$$

It is clear that $\{V(\varepsilon,\lambda): \varepsilon > 0,\ \lambda > 0\}$ is a filter-base on $X \times X$ and if $0 < \varepsilon_1 < \varepsilon$ and $0 < \lambda_1 < \lambda$ then $V(\varepsilon_1,\lambda_1) \subset V(\varepsilon,\lambda)$. Therefore this filter-base is clearly equivalent to (i.e. it generates the same filter as) the filter-base $\{V(\varepsilon,\varepsilon): \varepsilon > 0\}$ or even $\{V(\frac{1}{n}, \frac{1}{n}):$ n a positive integer$\}$. Similarly for each $x \in X$, the collection $\{N_x(\varepsilon,\lambda): \varepsilon,\lambda > 0\}$ is also a filter-base and is equivalent to the filter-base $\{N_x(\frac{1}{n},\frac{1}{n}):$ n a positive integer}. We note that for any $x \in X$

$$\bigcap_{n=1}^{\infty} N_x\left(\frac{1}{n}, \frac{1}{n}\right) = \{x\} .$$

A PPM-space $(X,F)$ induces a $T_1$ topology $\tau(F)$ on $X$ as follows: $U \in \tau(F)$ if for each $x \in U$, there exists some $\varepsilon > 0$ such that $N_x(\varepsilon,\varepsilon) \subset U$. In particular, each of the special types of PPM-spaces $(X,F)$ defined above induces a topology. A PPM-space $(X,F)$ is said to be topological if for each $x \in X$ and any $\varepsilon > 0$ the set $N_x(\varepsilon,\varepsilon)$ is a neighbourhood of $x$ in the topology $\tau(F)$.

Now we attempt to characterize the topologies induced by each of the special types of PPM-spaces defined above.

DEFINITION 2.1. *A topological space* $(X,\tau)$ *is said to be weakly first countable if there exists a function* $B : \omega \times X \to P(X)$ , *the power-set of* $X$, *such that for each* $x \in X$:

1) $\bigcap_{n<\omega} B(n,x) = \{x\}$,
2) $B(n+1,x) \subset B(n,x)$, *and*
3) $U \subseteq X$ *is open iff for each* $x \in X$, *some* $B(n,x)$ *is contained in* $U$.

*A function* $B$ *satisfying these conditions is called a wfc-function for* $(X,\tau)$.

THEOREM 2.1. *A topological space* $(X,\tau)$ *is weakly first countable (resp.* $T_1$ *and first countable) iff its topology is induced by a PPM-structure (resp. a topological PPM-structure) on* $X$.

P r o o f. Suppose $(X,\tau)$ is weakly first countable and let $B$ be a wfc-function for $(X,\tau)$. Without loss of generality we may assume that $B(0,x) = X$ for all $x \in X$. For any $x,y \in X$, $x \neq y$, let $n(x,y)$ denote the smallest integer $n$ such that $y \notin B(n,x)$ and $x \notin B(n,y)$. For any real number $r$ and any

$x,y \in X$, $x \neq y$ define

$$F_{xy}(r) = \begin{cases} 0 & \text{if } r \leq \frac{1}{n(x,y)} \\ 1 & \text{otherwise} \end{cases} .$$

Obviously, we define $F_{xx}(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ 1 & \text{if } r > 0 \end{cases}$ .

It is easy to verify that for any positive integer n, and any $x \in X$,

$$B(n,x) = N_x(\tfrac{1}{n}, \tfrac{1}{n}) .$$

Clearly, $\{F_{xy}\} = F$ is a PPM-structure with $\tau(F) = \tau$. If $(X,F)$ is $T_1$ and first countable, each $B(n,x)$ is open. Thus $N_x(\frac{1}{n}, \frac{1}{n})$ is open and $\tau(F)$ is topological.

If $\tau = \tau(F)$ where $F$ is a PPM-structure, put $B(n,x) = N_x(\frac{1}{n}, \frac{1}{n})$ and B is a wfc-function on $(X,\tau)$. Also, if $\tau(F)$ is topological, $N_x(\frac{1}{n}, \frac{1}{n}) \in \tau(F) = \tau$ and $\tau$ is $T_1$ and first countable.

**DEFINITION 2.2.** *A symmetric on a set* X *is a real-valued function* d *on* $X \times X$ *such that*

1) $d(x,y) \geq 0$ *and* $d(x,y) = 0$ *iff* $x = y$; *and*
2) $d(x,y) = d(y,x)$ .

Let d be a symmetric on a set X and for any $\varepsilon > 0$ and any $x \in X$, let $S(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$. We define a topology $\tau(d)$ on X by: $U \in \tau(d)$ iff for each $x \in U$, some $S(x,\varepsilon) \subset U$. A symmetric d is a semi-metric if for each $x \in X$ and each $\varepsilon > 0$, $S(x,\varepsilon)$ is a neighbourhood of x in the topology $\tau(d)$. A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X.

**THEOREM 2.2.** *A topological space* $(X,\tau)$ *is symmetrizable (semi-metrizable) iff its topology is induced by a symmetric (symmetric and topological) PPM-structure on* X.

P r o o f. Let $(X,\mathcal{F})$ be a symmetric PPM-space and let

$$d(x,y) = \begin{cases} \max\{\frac{1}{n} : y \notin N_x(\frac{1}{n},\frac{1}{n})\} \\ 0 \quad \text{if } y \in N_x(\frac{1}{n},\frac{1}{n}) \quad \text{for all } n. \end{cases}$$

Then d is a symmetric and $S(x,\frac{1}{n}) = N_x(\frac{1}{n},\frac{1}{n})$. Thus $\tau(\mathcal{F}) = \tau(d)$. If $\tau(\mathcal{F})$ is also topological, $S(x,\frac{1}{n}) = N_x(\frac{1}{n},\frac{1}{n})$ is an open neighborhood of x and d is a semi-metric with $\tau(\mathcal{F}) = \tau(d)$.

Now suppose $(X,\tau)$ is symmetrizable and let d be a symmetric on X such that $\tau(d) = \tau$. We may assume that $d(x,y) < 1$ for all $x,y \in X$. For any $x,y \in X$, define:

$$F_{xy}(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \leq d(x,y), \\ 1 & \text{if } \varepsilon > d(x,y). \end{cases}$$

It is easy to show that $(X,\mathcal{F})$ is a symmetric PPM-space. Moreover, for any $0 < \varepsilon < 1$, $S(x,\varepsilon) = N_x(\varepsilon,\varepsilon)$; and therefore $\tau(\mathcal{F}) = \tau(d)$. If d is a semi-metric, it is clear that $(X,\mathcal{F})$ is also topological. We note one other fact that will be used in the next theorem. If d is a metric IV holds and $(X,\mathcal{F})$ is a PM-space. For suppose $F_{xy}(\varepsilon) = 1$ and $F_{yz}(\delta) = 1$. Then $d(x,y) < \varepsilon$ and $d(y,z) < \delta$. Thus $d(x,z) < \varepsilon + \delta$ gives $F_{xz}(\varepsilon+\delta) = 1$.

The reader can compare Theorem 2.2 and its proof with the main theorem in [5].

Now we turn our attention to those PPM-structures which induce a metrizable topology. The result we have in this direction contrasts sharply with all known results of this type, in that we do not even postulate the existence of t-norm. The metrizability results known so for require the existence of a t-norm T satisfying the Menger triangle inequality $IV_m$ and some additional conditions on T like $\sup\{T(x,x) : 0 \leq x < 1 = 1\}$ (see [5], [7], and [8]).

THEOREM 2.3. *A topological space* $(X,\tau)$ *is metrizable if and only if there exists an H-structure on* X *which induces* $\tau$.

P r o o f. Suppose $(X,\tau)$ is metrizable and let d be a metric on X such that $\tau = \tau(d)$. For $x,y \in X$, define $F_{xy}$ as in Theorem 2.2. We showed in the proof of Theorem 2.2 that $F = F_{xy}$ satisfies I,II,III,IV and $\tau = \tau(F)$. It is easily seen that for any $0 < \varepsilon < 1$, the set $V(\varepsilon,\varepsilon)$ turns out to be the same set as $\{(x,y):d(x,y) < \varepsilon\}$. From this, it can be easily seen that $IV_h$ holds. Thus $(X,F)$ is an H-space.

Now suppose that $(X,F)$ is an H-space. Note that condition $IV_h$ can be restated as: For each $\varepsilon > 0$ there is a $\delta > 0$ such that $V(\delta,\delta) \circ V(\delta,\delta) \subseteq V(\delta,\delta)$. Therefore, the collection $\{V(\frac{1}{n},\frac{1}{n})$ :n a positive integer$\}$ is a countable base for a uniformity which induces the topology $\tau(F)$.Thus $(X,\tau(F))$ is metrizable.

REMARKS. (1) Not every PPM-structure inducing a metrizable topology is an H-structure.

(2) Condition $IV_h$ is necessary and sufficient to make the collection $\{V(\varepsilon,\lambda):\varepsilon,\lambda > 0\}$ a base for a uniformity.

(3) The Menger spaces with $\sup_{x<1} T(x,x) = 1$ are the spaces that have been studied extensively. These satisfy $IV_h$ [7], so the H-spaces contain them.

(4) The axioms for the H-spaces do not refer to a t-norm, thus they are easier to verify. The equilateral spaces of [7], and other examples, can easily be shown to satisfy $IV_h$. Another way of making a metric space into an H-space is to define $F_{xy}$ by

$$F_{xy}(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \le 0, \text{ and} \\ 1 - \dfrac{d(x,y)}{\varepsilon + d(x,y)} & \text{if } \varepsilon > 0. \end{cases}$$

(5) Given a symmetric PPM-structure $F$ on a set X, it is possibly to modify $F$ to obtain two symmetric PPM-structures $F_1$ and $F_2$ such that (i) $F_1$ satisfies axiom IV but $F_2$ does not, and (ii) $\tau(F) = \tau(F_1) = \tau(F_2)$. Therefore, each symmetrizable topology is induced by a PM-structure as well

as by a symmetric PPM-structure not satisfying axiom IV.

(6) The sufficiency of the condition in Theorem 2.3 is implicit in the proof of the metrization theorem in [7].

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REZIME

VEROVATNOSNE METRIČKE STRUKTURE: TOPOLOŠKA KLASIFIKACIJA

Data je karakterizacija topologija koje su indukovane nekim specijalnim tipovima pre-probabilističkih metričkih struktura.