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RANDOM NORMED STRUCTURES

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ABSTRACT

The axioms for a random normed space are simplified and other similar structures are defined and studied. In particular, it is shown that the metrizable topological linear spaces coincide with the linear spaces that admit a random H-structure.

1. INTRODUCTION

The introduction of the paper [2] in this journal gives all relevant definitions that are not given in this paper. The trivial distribution function H is defined by:

$$H(x) = 0 \quad \text{for} \quad x \leq 0, \text{ and } H(x) = 1 \quad \text{for} \quad x > 0.$$

DEFINITION 1.1. *X is a linear spaces over* R *or* C *and* F *is a function on* X *such that* $F(x) = F_x$ *is a distribution function. Let* $F = \{F_x : x \in X\}$ *and let* T *be a* t-*norm. The triple* (X, F, T) *is a random normed space if the following conditions hold:*

(R1) $F_x(0) = 0$ *for all* x *in* X.

(R2) $F_x = H$ *if and only if* $x = 0$.

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(R3) $F_{\lambda x}(u) = F_x(\frac{u}{|\lambda|})$ *for all* u, *all* $\lambda \neq 0$ *and all* x *in* X.

(R4) $F_{x+y}(u+v) \geq T(F_x(u), F_y(v))$ *for all* x,y *in* X *and all* $u \geq 0$, $v \geq 0$.

(R5) $T(u,v) \geq \max\{u+v-1,0\}$ *for all* u,v *in* $[0,1]$.

A random normed space is a Menger space if we put $G_{x,y} = F_{x-y}$. If the t-norm T is continuous, X is a T_2 topological linear space using the topology $t(F)$.

DEFINITION 1.2. *Suppose* X *is a linear space and for each* x *in* X *we have a distribution function* F_x *such that* (R1), (R2), (R3a) *and* (R3b) *hold.*

(R3a) *If* $0 < |\alpha| \leq 1$, $F_{\alpha x}(\varepsilon) \geq F_x(\varepsilon)$.

(R3b) *For any* x *and any* $\varepsilon > 0$, $F_{\alpha x}(\varepsilon) \to 1$ *as* $\alpha \to 0$.

The triple (X, F) *is a random pre-normed space (RPN-space).*

REMARKS. (1) (R3) implies (R3a) and (R3b).

(2) $F_x = F_{-x}$

(3) $N_x(\varepsilon,\lambda) = x + N_0(\varepsilon,\lambda)$.

P r o o f. For (R3a), $F_{\alpha x}(\varepsilon) = (\frac{\varepsilon}{|\alpha|}) \geq F_x(\varepsilon)$ since $\frac{\varepsilon}{|\alpha|} \geq \varepsilon$ when $0 < |\alpha| < 1$.

For (R3b), note that $F_{\alpha x}(\varepsilon) = F_x(\frac{\varepsilon}{|\alpha|}) \to 1$ as $\alpha \to 0$ since $\frac{\varepsilon}{|\alpha|} \to +\infty$.

(2) $F_{-x}(\varepsilon) \geq F_x(\varepsilon) = F_{(-1)(-x)}(\varepsilon) \geq F_{-x}(\varepsilon)$.

(3) $z \in N_x(\varepsilon,\lambda)$ if and only if $F_{x-z}(\varepsilon) > 1-\lambda$ if and only if $F_{0-(z-x)}(\varepsilon) > 1-\lambda$ if and only if $z-x \in N_0(\varepsilon,\lambda)$.

EXAMPLES. In each case, it is easy to verify (R1), (R2) and (R3) so that we have a RPN-space. The examples are given so the reader will not have to look elsewhere for them.

(1) Let X be a linear space. $G \neq H$ is a fixed distribution function with $G(0) = 0$. $x \to F_x$ where $F_0 = H$ and $F_x = G$ for $x \neq 0$.

(2) Let $(X, \| \ \|)$ be a normed linear space. G is a fixed distribution function with $G(0) = 0$ and $G \neq H$. Put $F_0 = H$ and for $x \neq 0$, put $F_x(r) = G(\frac{r}{\|x\|})$.

2. BASIC RESULTS

Unless stated otherwise, we will assume that $(X, \mathcal{F} = \{F_p : p \in X\})$ is a RPN-space. Let $G_{p,q} = F_{p-q}$ and $\mathcal{G} = \{G_{p,q} : p,q \in X\}$. Then $(X, \mathcal{G})$ is a symmetric PPM structure. The natural topology t(F) was defined earlier. t(F) is topological if for each $x \in X$ and any $\varepsilon > 0$, $N_x(\varepsilon, \varepsilon) \equiv N_x(\varepsilon)$ is a neighborhood of x in the topology t(F). It is natural to wonder when (X,t(F)) is a topological linear space.

THEOREM 2.1. *For each $\varepsilon, \lambda > 0$ we have*

(1) $N_0(\varepsilon, \lambda)$ *is balanced.*

(2) $N_0(\varepsilon, \lambda)$ *is absorbing.*

P r o o f (1) Let $x \in N_0(\varepsilon, \lambda)$ and suppose $0 < |\alpha| < 1$. Then $F_{\alpha x}(\varepsilon) \geq F_x(\varepsilon) > 1 - \lambda$ implies $\alpha x \in N_0(\varepsilon, \lambda)$. (2) Let $x \in X$. It suffices to show that there exists $\alpha \neq 0$ such that $\alpha x \in N_0(\varepsilon, \lambda)$. From (R3b), $F_{\alpha x}(\varepsilon) \to 1$ as $\alpha \to 0$. Thus we can choose $\alpha \neq 0$ such that $F_{\alpha x}(\varepsilon) > 1 - \lambda$.

THEOREM 2.2. *The following conditions are equivalent.*

(1) t(F) *is topological and* (X, t(F)) *is a topological linear space.*

(2) *For each* $\varepsilon > 0$, *there exists* $0 < \delta \leq \varepsilon$ *such that* $N_0(\delta) + N_0(\delta) \subset N_0(\delta)$.

(3) *For each* $\varepsilon > 0$, *there exists* $0 < \delta \leq \varepsilon$ *such that* $F_x(\delta) > 1-\delta$ *and* $F_y(\delta) > 1-\delta$ *implies* $F_{x+y}(\varepsilon) > 1-\varepsilon$.

(IV_h) *For each* $\varepsilon > 0$, *there exists* $0 < \delta \leq \varepsilon$ *such that* $G_{x,y}(\delta) > 1-\delta$ *and* $G_{y,z}(\delta) > 1-\delta$ *implies* $G_{x,z}(\varepsilon) > 1-\varepsilon$.

P r o o f. (1) implies (2). Assume (1) and let U^* denote all balanced neighborhoods of 0. Then $U^* \supset U = \{N_0(\varepsilon) : \varepsilon > 0\}$ since $t(F)$ is topological and each $N_0(\varepsilon)$ is balanced. Then ([7] page 96, Theorem 9.2) $N_0(\varepsilon) \in U^*$ implies there exists $V \in U^*$ such that $V + V \subset N_0(\varepsilon)$. Now $0 \in V$ and V is a neighborhood of 0 implies there exists $\delta > 0$ such that $N_0(\delta) \subset V$. We may assume $\delta \leq \varepsilon$. Then $N_0(\delta) + N_0(\delta) \subset N_0(\varepsilon)$.

(2) implies (1). Suppose (2) holds and consider U .

(a) Each U in U is balanced and absorbing.

(b) If $U \in U$ there exists $V \in U$ such that $V + V \subset U$.

(c) $N_0(\varepsilon_1) \cap N_0(\varepsilon_2) \supset N_0(\varepsilon)$ if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

We again use Theorem 9.2, page 96 of [7] to obtain a unique linear topology t having U as a base at 0. Clearly, $t = t(F)$.

(2) is equivalent to (3). This follows from the definition of $N_0(\varepsilon)$ and $F_p = F_{-p}$.

(3) is equivalent to (IV_h). This is immediate if you remember that $G_{x,y} = F_{x-y}$.

COROLLARY 2.1. *If* $t(F)$ *is topological and* $(X, t(F))$ *is a topological linear space, then* $(X, t(F))$ *is metrizable with an invariant metric* d *such that* $|\alpha| \leq 1$ *implies* $d(0,\alpha x) \leq d(0,x)$ *for all* $x \in X$.

P r o o f. It is clear from the proof of the theorem that $\{N_0(\frac{1}{n}) : n=1,2,3,\ldots\}$ is a countable base at 0 and

$(X,t(F))$ is a T_1 topological linear space. From [3], page 111, one obtains such a compatible metric.

DEFINITION 2.1. *If* (X,F) *satisfies* R1, R2, R3, R3a, R3b *and* IV_h, F *will be called a random* H-*structure for* X *and* (X,F) *will be referred to as a random* H-*space.*

Condition IV_h arose earlier in [2]. It is necessary and sufficient to make the collection $\{V(\varepsilon,\lambda):\varepsilon,\lambda > 0\}$ a base for a uniformity where

$$V(\varepsilon,\lambda) = \{(p,q):F_{p-q}(\varepsilon) \equiv G_{p,q}(\varepsilon) > 1-\lambda\} .$$

We have shown that every random H-space given rise to a metrizable topological linear space $(X,t(F))$. The next theorem establishes the converse result.

THEOREM 2.3. *Every metrizable topological linear space* (X,t) *admits a random* H-*structure* F *such that* $t = t(F)$.

P r o o f. Let d be a translation invariant compatible metric on X such that $|\alpha| \leq 1$ implies $d(0,\alpha x) \leq d(0,x)$. (See [3], page 111.) Define the distribution function F_p by

$$F_p(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \leq d(0,p), \text{ and} \\ 1 & \text{if } \varepsilon > d(0,p) . \end{cases}$$

Then we clearly have the following:

(R1) $F_p(0) = 0$ for every $p \in X$.

(R2) $F_p(\varepsilon) = 1$ for every $\varepsilon > 0$ if and only $p = 0$.

(R3a) $F_{\alpha p}(\varepsilon) \geq F_p(\varepsilon)$ if $|\alpha| \leq 1$. This holds since $d(0,\alpha p) \leq d(0,p)$.

For (R3b), note that as $\alpha \to 0$, $d(0,\alpha p) \to 0$ and so $F_{\alpha p}(\varepsilon) \to 1$ for each fixed ε and p.

(IV_h) Let $\varepsilon > 0$ be given and set $\varepsilon = \min\{\frac{\delta}{2}, 1\}$. Then $F_x(\delta) > 1-\delta$ and $F_y(\delta) > 1-\delta$ implies $F_x(\delta) = F_y(\delta) = 1$. Thus

$d(0,x) < \delta$ and $d(0,y) < \delta$. $d(x,x+y) < \delta$. Thus $d(0,x+y) \le d(0,x) + d(x,x+y) < 2\delta \le \varepsilon$. Hence $F_{x+y}(\varepsilon) = 1 > 1-\varepsilon$.

$t(\mathcal{F}) = t$. $N_0(\varepsilon,\lambda) = \{y : F_{0-y}(\varepsilon) = F_y(\varepsilon) > 1-\lambda\} = \{y : F_y(\varepsilon) = 1\} = \{y : d(0,y) < \varepsilon\} = S(0;\varepsilon)$. Also, $N_x(\varepsilon,\lambda) = x + N_0(\varepsilon,\lambda) = x + S(0;\varepsilon) = S(x;\varepsilon)$.

Consider the following condition:

(R4)´ $F_{p+q}(u+v) \ge \min\{F_p(u), F_q(v)\}$ for $u,v \ge 0$. It is easy to show that

(1) (R4)´ implies IV_h and

(2) (R4)´ is equivalent to $G_{x,z}(r+s) \ge \min\{G_{x,y}(r), G_{y,z}(s)\}$ where $G_{x,z} = F_{x-z}$. This condition arose in Theorem 2.1 of [1].

THEOREM 2.4. *If* (X,F) *satisfies* (R1), (R2), (R3), *and* (R4)´, $(X, t(\mathcal{F}))$ *is a locally convex metrizable topological linear space and* d *given by*

$$d(x,y) = \begin{cases} \sup\{\varepsilon : y \notin N_x(\varepsilon,\varepsilon),\ 0 < \varepsilon < 1\}, \\ 0 \quad \textit{if} \quad y \in N_x(\varepsilon,\varepsilon) \ \textit{for all} \quad \varepsilon > 0 \end{cases}$$

is a compatible translation invariant metric with $d(0,\lambda x) \le d(0,x)$, *if* $|\lambda| \le 1$. *Also* $d(x,y) < t$ *if and only if* $F_{x-y}(t) > 1-t$ *for* $0 < t < 1$.

P r o o f. Let $\mathcal{G} = \{G_{x,y} = F_{x-y} : x,y \in X\}$. Then $(X,\mathcal{G})$ is a symmetric PPM-space such that

$$G_{x,z}(r+s) \ge \min\{G_{x,y}(r), G_{y,z}(s)\} .$$

From Theorem 2.1 of [1], d is a compatible metric for $t(\mathcal{G}) = t(\mathcal{F})$, $S(x,\varepsilon) = N_x(\varepsilon,\varepsilon)$, and $d(x,y) < t$ if and only if $F_{x-y}(t) > 1-t$.

(1) $d(x+y, z+y) = d(x,z)$.

Since $N_x(\varepsilon) = \{y : F_{x-y}(\varepsilon) > 1-\varepsilon\}$, $d(0,x-z) = \sup\{\varepsilon : F_{0-(x-z)}(\varepsilon) \le 1-\varepsilon\} = \sup\{\varepsilon : F_{x-z}(\varepsilon) \le 1-\varepsilon\} = d(x,z)$.

Now $d(x+y,z+y) = d(0,x+y-[z+y]) = d(0,x-z) = d(x,z)$.

(2) $d(0,\lambda x) \leq d(0,x)$ if $|\lambda| \leq 1$.

If $|\lambda| \leq 1$, $F_p(\varepsilon) \leq F_{\lambda p}(\varepsilon)$ by (R3a). Thus $d(0,\lambda p) = $ $= \sup\{\varepsilon : F_{\lambda p}(\varepsilon) \leq 1-\varepsilon\} \leq \sup\{\varepsilon : F_p(\varepsilon) \leq 1-\varepsilon\} = d(0,p)$.

(3) $N_0(\varepsilon)$ is convex.

Let $y_1, y_2 \in N_0(\varepsilon)$ and $0 < \lambda < 1$. Then $F_{y_1}(\varepsilon) > 1-\varepsilon$ and $F_{y_2}(\varepsilon) > 1-\varepsilon$. Now if $F_{\lambda y_1}(\lambda\varepsilon) \leq F_{(1-\lambda)y_2}([1-\lambda]\varepsilon)$, we have $F_{\lambda y_1+(1-\lambda)y_2}(\lambda\varepsilon+[1-\lambda]\varepsilon) \geq \text{Min}\,\{F_{\lambda y_1}(\lambda\varepsilon), F_{(1-\lambda)y_2}([1-\lambda]\varepsilon)\} = $ $F_{\lambda y_1}(\lambda\varepsilon) = F_{y_1}(\varepsilon) > 1-\varepsilon$. Thus $[y_1 + (1-\lambda)y_2] \in N_0(\varepsilon)$.

COROLLARY 2.2. *Suppose (X,F) is a RPN-space and the t-norm T given by $T(a,b) = \min\{a,b\}$ satisfies* (R4). *Then the conclusions of Theorem 2.4. hold.*

If one is in the setting of Theorem 2.4, we have $d(x,y) < t$ if and only if $F_{x-y}(t) > 1-t$ for $0 < t < 1$. This provides a means of translating many fixed point theorems and other concepts from metric spaces. See [1] for the details.

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REZIME

SLUČAJNE NORMIRANE STRUKTURE

Ispitane su slučajne normirane strukture i pokazano je da se metrizabilni linearni topološki prostori poklapaju sa linearnim prostorima koji dopuštaju H-strukturu.