Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 14,1 (1984)
Review of Research Faculty of Science-University of Novi Sad, Volume 14,1(1984)

CLASSIFICATION OF P_3 AND THE ENUMERATION OF BASES OF P_3

Ivan Stojmenović

Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

In [3] Miyakawa proved the existence of 418 classes of three-valued logical functions. In this paper it is proved that the number of classes of functions in P_3 is 406 (and not 418). This paper contains data on the number of bases and pivotal incomplete sets of any rank and an algorithm for enumeration which is different from that in [4] (The data given in [4] is incorrect because the number of classes is incorrect).

INTRODUCTION

Let $E_3 = \{0,1,2\}$ and $P_3^{(n)}$ be the set of all functions from the cartesian product $(E_3)^n$ into E_3 $(n > 0)$, and P_3 be the union $\bigcup_{n=1}^{\infty} P_3^{(n)}$.

A composition of functions of the system $\{f_1, f_2, \ldots, f_s, \ldots\} \subset P_3$ is:

a) any function which can be obtained by the change of variables

b) any function which can be obtained by the change of

AMS Mathematics subject classification (1980): Primary 03B50; Secondary 05A15
Key words and phrases: Three-valued logic algebra, precomplete sets, bases, pivotal sets, enumeration, classes of functions, classification.

variables,equalizing of arguments and addition of a dummy argument from the function $F(F_1(y_{11},\dots,y_{1m_1}),\dots,F_p(y_{p1},\dots \dots,y_{pm_p}))$, where $F(y_1,\dots,y_p)$ is a composition of functions of the system and $F_i(y_{i1},\dots,y_{im_i})$ is a composition of functions of the system or y_i $(i=1,\dots,p)$.

A set of function in P_3 is complete if every function in P_3 can be represented as a composition of the elements of the set. A set of functions in P_3 is nonredundant if its elements can be represented as a composition of the other elements of the set. A base is a nonredundant set. A precomplete set is a maximal incomplete set.

We shall recall some notations of the functions preserving a h-ary relation ρ $(1 \le h \le 3)$. We denote it by a matrix, i.e. $\rho^t \subset E_3^h$. Then for n-ary vectors $a_1,\dots,a_h$ $(a_i=(a_{i1},\dots,a_{in})\in E_3^n)$,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \rho \iff \text{for all } i, \begin{pmatrix} a_{1i} \\ \vdots \\ a_{hi} \end{pmatrix} \in \rho \quad .$$

The set of functions preserving ρ (denoted by Polρ) is defined by

$$\mathrm{Pol}\,\rho = \{f \mid \begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \rho \Rightarrow \begin{pmatrix} f(a_1) \\ \vdots \\ f(a_n) \end{pmatrix} \in \rho\} \; .$$

THEOREM 1. ([1]) *The following eighteen subsets of* P_3 *are precomplete, and there is no other precomplete set.*

1) $T = \mathrm{Pol}(\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid \mathrm{card}\{a,b,c\} \le 2\})$ (Here the notation card A for a set A denotes the cardinality of A)

2) $L = \mathrm{Pol}(\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid c \equiv 2(a+b) \pmod 3\})$

3) $S = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

4) $M_1 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$ 5) $M_2 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$

6) $M_0 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 & 2 \end{pmatrix}$ 7) $U_2 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$

8) $U_0 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{pmatrix}$ 9) $U_1 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix}$

10) $B_0 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 & 0 \end{pmatrix}$ 11) $B_1 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 1 \end{pmatrix}$

12) $B_2 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 & 1 & 2 \end{pmatrix}$ 13) $T_0 = \mathrm{Pol}(0)$

14) $T_1 = \mathrm{Pol}(1)$ 15) $T_2 = \mathrm{Pol}(2)$

16) $T_{01} = \mathrm{Pol}(0\ 1)$ 17) $T_{12} = \mathrm{Pol}(1\ 2)$

18) $T_{02} = \mathrm{Pol}(0\ 2)$

It is known that the functions in P_3 can be classified by using the precomplete sets so that we can discuss completeness in terms of these classes instead of the functions themselves. More precisely, if a set is complete, then replacing the functions in the set by any functions in the corresponding classes yields another complete set.

In this paper we shall prove that the number of classes of functions in P_3 is 406 (and not 418 [3]).

The number of classes of functions in a set is called the rank of the set.

Each class is represented by its characteristic vector (hereafter simply vector), an 18 bit string, where the bit or 1 at the i-th position denotes that the class is, resp., is not, included in the i-th precomplete set. The completeness of a set of given classes can be tested by examining their vectors, namely the criterion for completeness is the following:

1) bitwise OR for the vector corresponding of the classes results 11...1 (completeness).

Redundancy of a set of given classes is related to the following condition:

2) for each class of the set, bitwise OR for all the vectors of the classes except the class is not equal to one for all the vectors of the classes (pivotalness).

In other words, the last condition is equivalent to saying that for every class there is at least one precomplete set in which the class is not included and all other classes are included. We call a set pivotal if it satisfies condition 2). It is easy to see that a pivotal set is nonredundant. Hence a complete set is a base if and only if it is a pivotal set. We can say that a base corresponds to a minimal cover of 11... ...1, and a pivotal set corresponds to a minimal cover of some binary vector (in which some o's may occur).

This paper contains data on the number of bases and pivotal incomplete sets of any rank and an algotithm for enumeration which is different from that in [4]. (The data given in [4] is incorrect because the number of classes is incorrect).

2. CLASSIFICATION OF P_3

In [3] a proof is given that there exist 418 types of functions of three-valued logic.

Let $D'(0,1) = \{f \mid \text{range } f \subset \{0,1\}\} \setminus \{0,1\}$ and let us define $D'(1,2)$, $D'(0,2)$ analogously.

Also let $A(i,j,k) = \{f \mid f(0,\dots,0) = i,\ f(1,\dots,1) = j,\ f(2,\dots,2) = k \text{ and } f \in D'(0,1)\}$.

Using the notation, $D'(0,1)$ can be represented as $D'(0,1) = \sum_{i,j,k=o}^{1} A(i,j,k)$, where the summation is the direct sum of sets.

On the occasion of the classification of the precomplete set $T = D'(0,1) \cup D'(1,2) \cup D'(0,2) \cup P_3^{(1)}$ the set $D'(0,1)$ was divided into the sets $A(i,j,k)$ and then the classes of functions for every set $A(i,j,k)$ were determined. In [3] it is not remarked that the functions which are in different sets $A(i,j,k)$ can be in the same class P_3 (for example functions denoted by f.5.4 and f.5.6 in [3]). This is the reason why 12 classes are enumerated two times.

These are the following pairs of classes: (in the appendix of [3] denoted by #)
(188,189),(190,191),(192,193),(240,241),(242,243),(244,245),(246,247),(250,251),(252,253),(300,301),(304,305),(310,311).

So, the number of these classes in P_3 is 406. The complete list of classes is given in [3] (without the superfluous classes) and in [7].

3. ENUMERATION OF THE BASES OF THREE-VALUED LOGICAL FUNCTIONS

The enumeration of bases is given in [4]. But in this paper the incorrect number of classes is used and so some of the results in [4] are not corect.

Let A(1),...,A(418) be the vectors of the corresponding classes. The algorithm, used in [4], for the given rank r, checked for every set of vectors is this set of vectors the base (or pivotal incomplete set). The vectors, with the length greater than 19-r (by determination of bases) and greater than 18-r (for pivotal sets), are not included by the determination of bases and pivotal incomplete sets of the rank r.

Here we shall use another algorithm, which is, in our opinion, more effective. By our algorithm we can find, at the same time, all the pivotal sets of all possible ranks, and it is not necessary to investigate the maximal rank of bases and pivotal sets.

Let b_i be the number of bases and p_i the number of pivotal sets of the rank i, and A(1),...,A(m) the vectors of the corresponding classes. Let us denote by n the number of almost complete sets. In the case P_3 is n=18, m=406.

Step 1. $j_1 = 0$, $b_i = p_i = 0$ $(1 \leq i \leq n)$.

Step 2. $s = 1$, $j_1 = j_1 + 1$, $p_1 = p_1 + 1$.

Step 3. If $j_s = m$ go to step 7. In the opposite case take $k=0$ and go to the next step.

Step 4. $k = k+1$, $j_{s+1} = j_s + k$. If $A(j_1),\dots,A(j_s)$, $A(j_{s+1})$ is not pivotal set, go to step 5. In the opposite case set $s=s+1$. If $A(j_1),\dots$ $\dots,A(j_s)$ is the base the set $b_s = b_s + 1$ and go to step 6, and if $A(j_1),\dots,A(j_s)$ is not the base, take $p_s = p_s + 1$ and go to step 3.

Step 5. If $j_s + k < m$, go to step 4. In the opposite case go to the next step.

Step 6. If $s=1$, go to step 2. If $s > 1$, set $s=s-1$, take $k = j_{s+1} - j_s$ and go to step 4.

Step 7. If $s=1$, print b_i, p_i and finish the algorithm. If $s \neq 1$ take $s=s-2$. If $s=0$, go to step 2. If $s \neq 0$ $k = j_{s+1} - j_s$ and go to step 4.

In the check of pivotality we use two auxiliary sequences c and r. c_i $(1 \le i \le n)$ is the number of units in the i-th column in the vectors $A(j_1),\dots,A(j_s)$. The sequence r_i $(1 \le i \le s)$ has the following property: the r_i-th coordinate of the vector $A(j_i)$ is equal to 1 nad the r_i-th coordinate of the vector $A(j_t)$, $1 \le t \le s$, $t \neq i$ is equal to 0.

Here we summarize the enumeration results.
The computer DELTA 340 of the Institute of Mathematics, University of Novi Sad is used.

rank	bases	pivotal incomplete sets	bases which contain the constant functions {0,1,2}	pivotal incomplete sets which contain the constant functions {0,1,2}
1	1	404	0	3
2	8265	60335	0	3
3	794256	1418970	0	1
4	4612601	2677899	2	54
5	810474	176187	633	495
6	14124	1368	756	237
7	0	9	0	9
Σ	6239721	4335172	1391	802

Corollary 1. The maximal rank of bases of P_3 is 6.

Corollary 2. The maximal rank of pivotal sets is 7.

Corollary 3. The number of bases of P_3 is exactly 6239721.

Corollary 4. The number of pivotal incomplete sets is exactly 4335172.

Corollary 5. The number of bases which contain the constant functions {0,1,2} is exactly 1391.

Corollary 6. The number of pivotal incomplete sets which contain the constant functions {0,1,2} is exactly 802.

REFERENCES

[1] Яблонский С.В., Функциональные построения в к-значных логиких, Труды МИ АН СССР, Т.51, Москва, 1958, 5-142

[2] Lau D., *Submaximale Klassen von P_3, Electronische Informationsverarbeitung und Kybernetik EIK 18(1982) 4/5, 227-243.*

[3] Miyakawa M., *Functional Completeness and Structure of Three-valued Logic I-Classification of P_3, Researches of Electrotechnical Laboratory No. 717, 1971, 1-85.*

[4] Miyakawa M., *Enumeration of Bases of Three-valued Logical Functions, Colloquia Mathematica Societatis Janos Bolyai 28, Szeged, North Holland, 1981, 469-487.*

[5] Miyakawa M., *Enumeration of Bases of a Submaximal Set of Three-valued Logical Functions, Rostocker Mathematisches Kolloquium, Heft **19, Wilhelm-Pieck Universität Rostock, 1982, 49-67.***

[6] Rosenberg I.G., *Completeness Properties of **Multiple-valued** Logic Algebras, in: Computer Science and Multiple-valued Logic (ed.D.C.Rine), North-Holland, Amsterdam, 1977, pp.144-186.*

[7] Stojmenović I., *Simetrične funkcije dvoznačne i troznačne logike, Master's Thesis, Novi Sad, 1983.*

Received by the editors September 6, 1983.

REZIME

KLASIFIKACIJA P_3 I PREBROJAVANJE BAZA P_3

U radu [3] Miyakawa je dokazao da postoji 418 tipova funkcija troznačne logike. U ovom radu je pokazano da postoji greška u [3] i da je broj tipova 406. Zbog toga su i podaci o broju tipova baza i tipova pivotalnih nekompletnih skupova troznačne logike dobijeni u [4] pogrešni. Tačni podaci su dati u ovom radu korišćenjem verovatno efikasnijeg algoritma.