Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 14,1 (1984)
Review of Research Faculty of Science-University of Novi Sad, Volume 14,1(1984)

ENUMERATION OF THE BASES OF THREE--VALUED MONOTONE LOGICAL FUNCTIONS

Ivan Stojmenović
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br. 4, Jugoslavija

ABSTRACT

In this paper the number of classes of functions, the number of bases and the number of the pivotal noncomplete sets for the set of the monotone functions of three-valued logic are determined.

1. INTRODUCTION

The set of three-valued logical functions, i.e., the union of all the functions $\{f:\{0,1,2\}^n \to \{0,1,2\}\}$ for $n=0,1,2,\ldots$ is denoted by P_3.

A composition of functions of the system $\{f_1, f_2, \ldots, f_s, \ldots\} \subset P_3$ is:

a) any function which can be obtained by the change of variables
b) any function which can be obtained by the change of variables, the equalizing of arguments and the addition of the dummy (fixed) argument from the function $F(F_1(y_{11},\ldots,y_{1m_1}),\ldots, F_p(y_{p1},\ldots,y_{pm_p}))$, where $F(y_1,\ldots,y_p)$ is a composition of functions of the system and $F_i(y_{i1},\ldots y_{im_i})$ is a composition of functions of the system or y_i $(i=1,\ldots,p)$.

A subset F of P_3 is said to be closed if it does not yield a function which is not in F by means of compositions among functions in F. For closed sets F and H such that $F \subset H$ (proper inclusion), F is H-maximal if there is no closed set G such that $F \subset G \subset H$. A subset of functions in H (for closed sets $H \subset P_3$) is

---

AMS Mathematics subject classification (1980): Primary 03B50, Secondary 05A15
Key words and phrases: Three-valued logic algebra, precomplete sets, bases, pivotal sets, enumeration, classes functions, monotone functions.

complete if every function in H can be represented as a composition of the elements of the set. A subset of H is nonredundant if none of its elements can be represented as a composition of the other elements of the set. A base is a nonredundant complete set.

There is a well-known approach to the functional completeness problem. If we know all the maximal sets, then a subset F is complete if F is not included entirely in every maximal set [1]. Further, there is a straightforward procedure for enumerating the bases. We classify the set into equivalent classes so that we can discuss the completeness property in terms of these classes instead of each function. More precisely, if a set is complete, then replacing the functions in the set by any functions in the corresponding equivalent classes yields another complete set. The classification is one which uses all maximal sets. The next step is to enumerate the bases using this classification, and this completes a cycle of work on the functional completeness problem.

We shall recall some notations of functions preserving a h-ary relation ρ . We denote it by a matrix, i.e. $\rho^t \subset E_3^h$, $E_3 = \{0,1,2\}$.

Then for n-ary vectors $a_1, a_2, \ldots, a_h$ $(a_i \in E_3^n)$,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \rho \Leftrightarrow \text{for all } i, \begin{pmatrix} a_{1i} \\ \vdots \\ a_{hi} \end{pmatrix} \in \rho .$$

Then the set of functions preserving ρ (denoted Polρ) is defined by

$$\mathrm{Pol}\ \rho = \{ f \mid \begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \rho \Rightarrow \begin{pmatrix} f(a_1) \\ \vdots \\ f(a_h) \end{pmatrix} \in \rho \} .$$

THEOREM 1 ([1]) *The following eighteen subsets of* P_3 *are* P_3*-maximal, and there is no other* P_3*-maximal set.*

1. $T_0=\mathrm{Pol}(\ 0\ )$ 2. $T_1=\mathrm{Pol}(\ 1\ )$ 3. $T_2=\mathrm{Pol}(\ 2\ )$

4. $T_{01}=\mathrm{Pol}(\ 0\ 1\ )$ 5. $T_{12}=\mathrm{Pol}(\ 1\ 2\ )$ 6. $T_{02}=\mathrm{Pol}(\ 0\ 2\ )$

7. $B_0=\mathrm{Pol}\begin{pmatrix} 0&1&2&0&1&0&2 \\ 0&1&2&1&0&2&0 \end{pmatrix}$ 8. $B_1=\mathrm{Pol}\begin{pmatrix} 0&1&2&1&0&1&2 \\ 0&1&2&0&1&2&1 \end{pmatrix}$

9. $B_2=\mathrm{Pol}\begin{pmatrix} 0&1&2&2&0&2&1 \\ 0&1&2&0&2&1&2 \end{pmatrix}$

10. $U_2=\mathrm{Pol}\begin{pmatrix} 0&1&2&0&1 \\ 0&1&2&1&0 \end{pmatrix}$ 11. $U_1=\mathrm{Pol}\begin{pmatrix} 0&1&2&0&2 \\ 0&1&2&2&0 \end{pmatrix}$

12. $U_0=\mathrm{Pol}\begin{pmatrix} 0&1&2&1&2 \\ 0&1&2&2&1 \end{pmatrix}$

13. $M_0=\mathrm{Pol}\begin{pmatrix} 0&1&2&2&2&0 \\ 0&1&2&0&1&1 \end{pmatrix}$ 14. $M_1=\mathrm{Pol}\begin{pmatrix} 0&1&2&0&0&1 \\ 0&1&2&1&2&2 \end{pmatrix}$

15. $M_2=\mathrm{Pol}\begin{pmatrix} 0&1&2&1&1&2 \\ 0&1&2&2&0&0 \end{pmatrix}$

16. $L=\mathrm{Pol}(\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_3^3\ ;\ c \equiv 2(a+b)(\mathrm{mod}\ 3)\})$

17. $S=\mathrm{Pol}\begin{pmatrix} 0&1&2 \\ 1&2&0 \end{pmatrix}$

18. $T=\mathrm{Pol}(\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in E_3^3\ ;\ \mathrm{card}\ \{a,b,c\} \leq 2\})$

Here the notation card A for a set A denotes the cardinality of A.

The intersection $X_1 \cap X_2 \cap \ldots \cap X_k$ shall be denoted by $X_1X_2\ldots X_k$. For the set M_1 of monotone functions in respect to the order $0<1<2$, the following theorem is proved.

THEOREM 2. *(Machida, [3]) The following thirteen subsets of M_1 are M_1-maximal, and there is no other M_1-maximal set:*

1. T_0M_1 2. T_2M_1 3. $T_{01}M_1$ 4. $T_{12}M_1$ 5. $T_{02}M_1$ 6. B_0M_1

7. B_1M_1 8. B_2M_1 9. $U_0\ M_1$ 10. $U_2\ M_1$ 11. TM_1

12. $M_{min}=\{f \in P_3 | f=\min(\ t_1(x_1),\ldots,t_n(x_n))\}$,

13. $M_{max}=\{f \in P_3 | f=\max(\ t_1(x_1),\ldots,t_n(x_n))\}$.

$t_1(x_1),\ldots,t_n(x_n)$ are the unary monotone functions. Let

t_{abc} be such function that $t_{abc}(0)=a$, $t_{abc}(1)=b$, $t_{abc}(2)=c$.

Then $t_i \in U=\{t_{000}, t_{001}, t_{002}, t_{011}, t_{012}, t_{022}, t_{111}, t_{112}, t_{122}, t_{222}\}$.

The value of the function min(max) is the least (biggest) value among the arguments.

In this paper we shall consider a particular maximal set M_1 and classify it by using Machida's thirteen submaximal sets. Each class is represented by its characteristic vector, a 13 bit string, where the bit 0 or 1 at the i-th position denotes that the class is, resp., is not, included in the i-th maximal set. The completeness of a set of given classes can be tested by examining their vectors, namely the criterion for completeness is the following:

1) bitwise OR for the vectors corresponding to the classes results 11...1 (completeness).
   The redundancy of a set of given classes is related to the following condition:
2) for each class of the set, bitwise OR for all the vectors of the classes except the class is not equal to one for all the vectors of the classes (pivotalness).

In other words, the last condition is equivalent to saying that for every class there is at least one maximal set in which the class is not included and all other classes are included. We call a set pivotal if it satisfies condition 2). It is easy to see that a pivotal set is nonredundant. Hence a complete set is a base if and only if it is a pivotal set. We can say that a base corresponds to a minimal cover of 11...1, and a pivotal set corresponds to a minimal cover of some binary vector (in which some 0's may occur).

In this paper we shall prove that the number of nonempty classes is 88 in contrast to the possible 8192 classes.

We show that there exist exactly 118744 M_1-bases (bases of M_1) and 152651 M_1-pivotal noncomplete sets.

2. THE CLASSIFICATION OF THE SET M_1

The classes of functions in P_3 are determined in papers [4], [7], [8]. According to [4] there exist 418 classes but some of these classes are not different. The number of different classes are 406 ([7],[8]).

The sets $T_0, T_2, T_{01}, T_{12}, T_{02}, B_0, B_1, B_2, U_0, U_2$ and T are, as is the set M_1 also, maximal sets in P_3. Among 406 classes of three-valued logic, for 48 classes the component of the characteristic vector which corresponds to the set M_1 is 0. These classes are denoted by *1-*48 in the second column of the table at the end of the paper. In this way a classification of the functions of the set M_1 is obtained in respect to the first 11 M_1-maximal sets. The complete classification of the set M_1 will be obtained if we determine for every set $M_{min}M_{max}$, $M_{min}\bar{M}_{max}$, $\bar{M}_{min}M_{max}$, $\bar{M}_{min}\bar{M}_{max}$ whether it contains the functions of the classes *1-*48.

3. THE CLASSIFICATION OF THE SET $M_{min}M_{max}$

THEOREM 3. *The set $M_{min}M_{max}$ contains only monotone functions which depends on at most only one variable, i.e. if $f \in M_{min}M_{max}$ and f do not contain the dummy variable then $f \in U$.*

P r o o f: Suppose $f \in M_{min}M_{max}$ and f depends on at least 2 variables, then $f(x_1,\dots,x_n)=\min(t_1'(x_1),\dots,t_n'(x_n))=$ $=\max(t_1''(x_1),\dots,t_n''(x_n))$, $t_i', t_i'' \in U$. Because $f(x_1,\dots,x_n)\neq 1$ and $f \in M_1$, one of two cases is satisfied:

a) $f(0,\dots,0)=0$ b) $f(2,\dots,2)=2$

Consider the case a) (analogously for case b)): $f(0,\dots,0)=0$, so there exists i such that $t_i'(0)=0$. Then $f(x_1,\dots,x_{i-1},0,x_{i+1},\dots$ $\dots x_n)=\min(t_1'(x_1),\dots t_{i-1}'(x_{i-1}), t_i'(0), t_{i+1}'(x_{i+1}),\dots,t_n'(x_n))=$ $=0=\max(t_1''(x_1),\dots,t_{i-1}''(x_{i-1}), t_i''(0), t_{i+1}''(x_{i+1}),\dots,t_n''((x_n))$, hence $t_i''(0)=0$ and $t_1''(x_1)=\dots=t_{i-1}''(x_{i-1})=t_{i+1}''(x_{i+1})=\dots=t_n''(x_n)=0$.

Function f does not depend on $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$, hence f depends on at most only one variable. This is a contradiction.

Considering the functions of the set U, the classes of the set $M_{min}M_{max}$ are:

$*47(t_{000})$, $*19(t_{001})$, $*37(t_{002})$, $*40(t_{011})$, $*48(t_{012})$

$*39(t_{022})$, $*46(t_{111})$, $*38(t_{112})$, $*18(t_{122})$, $*45(t_{222})$.

On the occasion of classification of the set $\bar{M}_{min}\cup\bar{M}_{max}$, for functions which do not contain the dummy variable, the functions t_{000} , t_{111}, t_{222} are not used.

LEMMA 1. ([4]), theorem 4.4) $B_0B_1B_2=\{0,1,2,x_i\ (i=1,2,\dots)\}$.

From Lemma 1, the classes *45-*48 contain only the functions $0,1,2,x_i$ of the set $M_{min}M_{max}$. Hence, classes *45-*48 do not contain functions of the sets $M_{min}\bar{M}_{max}$, $\bar{M}_{min}M_{max}$, $\bar{M}_{min}\bar{M}_{max}$.

4. THE CLASSIFICATION OF THE SET $M_{min}\bar{M}_{max}$

From Theorem 1 it follows that the functions from the set M_{min} with $n\geq 2$ variables are not in the set M_{max}.

Let $f(x_1,\dots,x_n)=\min(t_1(x_1),\dots,t_n(x_n))\in M_{min}, n\geq 2$, $t_i\in\{t_{001}, t_{002},t_{011},t_{012},t_{022},t_{112},t_{122}\}$.

We shall consider two cases.

1^o there exists i so that $t_i\in\{t_{001},t_{011}\}$.

It is easy to check that $f\in T_0\bar{T}_2T_{01}\bar{T}_{02}B_0B_1\bar{B}_2U_2\ T$.

LEMMA 2. *For* $f\in M_{min}$ *is* $f\in T_{12}$ <=> *for every* $i, 1\leq i\leq n$, $t_i\notin\{t_{001},t_{002}\}$ <=> $f\in U_0$.

P r o o f: If, for some i, $t_i=t_{001}$ or $t_i=t_{002}$ then $f(1,\dots,1)=0$, hence $f\in\bar{T}_{12}\bar{U}_0$ .From $t_i\in\{t_{011},t_{012},t_{022},t_{122},t_{112}\}$ it follows that $t_i(1)\geq 1, t_i(2)\geq 1$, and so $f\in T_{12}U_0$.

The functions $\min(t_{001}(x_1),t_{011}(x_2))$ and

$\min(t_{011}(x_1), t_{012}(x_2))$

are examples of possible classes *19 and *40 in the case 1^o.

2^o For every i is $t_i \in \{t_{002}, t_{012}, t_{022}, t_{112}, t_{122}\}$.

If f satisfies this condition, then we have the following nine lemmas.

LEMMA 3. $f \in T_2$.

The proof is obvious.

LEMMA 4. $f \in T_0$ *if and only of there exists* i *so that* $t_i \in \{t_{002}, t_{012}, t_{022}\}$.

P r o o f: Since $t_i \in \{t_{112}, t_{122}\}$ for every i, $1 \le i \le n$ if and only if $f(0,\dots,0)=1$, the proof is complete.

LEMMA 5. $f \in T_{01}$ *if and only if there exists* i *so that* $t_i \in \{t_{002}, t_{012}, t_{112}\}$.

P r o o f: Since $t_i \in \{t_{022}, t_{122}\}$ for every i, $1 \le i \le n$ if and only if $f(1,\dots,1)=2$ the proof is complete.

LEMMA 6. $f \in T_{12} \iff f \in U_0 \iff$ *for every* i, $t_i \ne t_{002}$.

P r o o f: The proof follows from Lemma 1 and condition 2^o.

LEMMA 7. $f \in T_{02} \iff f \in B_0 \iff$ *for every* i $(1 \le i \le n)$, $t_i \in \{t_{002}, t_{012}, t_{022}\}$.

P r o o f: Let us denote $\underbrace{a\dots a}_{k}$ by a^k, $a \in \{0,1,2\}$. Let $t_i = t_{112}$ or $t_i = t_{122}$. Then $f(2^{i-1}, 0, 2^{n-i}) = 1$, and so it follows that $f \notin T_{02}$. Since $f(2,\dots,2)=2$ we have $f \notin B_0$.
Let $t_i \in \{t_{002}, t_{012}, t_{022}\}$ for every i, $1 \le i \le n$. From $f(x_1,\dots,x_n)=1$ it follows that there exists i so that $t_i = t_{012}$ and $x_i = 1$. From this it follows that $f \in T_{02}$. Since $f(y_1,\dots,y_n)=2$ implies $y_i = 2$, we have that $f \in B_0$.

LEMMA 8. $f \in U_2$ *if and only if for every* i $(1 \le i \le n)$, $t_i \in \{t_{002}, t_{012}, t_{112}\}$.

P r o o f: If, for every i, $t_i \in \{t_{002}, t_{012}, t_{112}\}$, then $f(x_1,\dots,x_n)=2$ if and only if $x_1=\dots=x_n=2$ and so $f \in U_2$. If there exists i so that $t_i \in \{t_{022}, t_{122}\}$ and $1 \le i \le n$ then $f(2^{i-1}, 1, 2^{n-i})=2$, $f(2^{i-1}, 0, 2^{n-i}) \in \{0,1\}$ and so $f \notin U_2$.

LEMMA 9. $f \in T$ *if and only if* $f \notin T_0$ *or* $t_i \in \{t_{002}, t_{022}\}$ *for every* i, $1 \le i \le n$.

P r o o f: A function $f \in T$ with $n \ge 2$ variables does not take only one of the values $0,1,2$. Since $f(2,\dots,2)=2$ f must take the value 2. Hence $f(0,\dots,0)\ne 0$. Since $f \in M_1$ from this it follows that $f(x_1,\dots,x_n) \ne 0$ and $f \in T$. The last possibility is that $f(x_1,\dots,x_n)\ne 1$. This condition is satisfied if $t_i \in \{t_{002}, t_{022}\}$ for every i, $1 \le i \le n$. If there exists i $(1 \le i \le n)$ so that $t_i = t_{ab2}$ and $a=1$ or $b=1$, then $f(2^{i-1}, c, 2^{n-i})=1$, where $c=0$ for $a=1$ and $c=1$ for $b=1$.

LEMMA 10. $f \in B_1$ *if and only if* $t_i \in \{t_{012}, t_{112}, t_{122}\}$ *for every* i, $1 \le i \le n$.

P r o o f: Let $t_i \in \{t_{012}, t_{112}, t_{122}\}$ for every i, $1 \le i \le n$. From $f(x_1,\dots,x_n)=0$ it follows that there exists i so that $t_i = t_{012}$ and $x_i=0$. On the other hand, from $f(x_1,\dots,x_n)=2$ it follows that $x_i=2$, and so $f \in B_1$.

If there exists i so that $t_i = t_{002}$ then $f(2^{i-1}, 1, 2^{n-i})=0$ and $f(2,\dots,2)=2$, which implies $f \notin B_1$. If $t_i = t_{022}$ for some i, $1 \le i \le n$, then $f(2^{i-1}, 1, 2^{n-i})=2$ and $f(2^{i-1}, 0, 2^{n-i})=0$ and so $f \notin B_1$.

LEMMA 11. $f \in B_2$ *if and only if* $t_i = \{t_{112}, t_{122}\}$ *for every* i, $1 \le i \le n$, *or* $t_i \in \{t_{002}, t_{022}\}$ *for every* i, $1 \le i \le n$.

P r o o f: If $t_i \in \{t_{112}, t_{122}\}$ for every i, then $f(x_1,\dots,x_n)\neq 0$, and so $f \in B_2$. If $t_i \in \{t_{002}, t_{022}\}(1\le i\le n)$ then $f(x_1,\dots,x_n)\neq 1$ and so $f \in B_2$.

If the condition in this lemma is not satisfied then there are i and j so that $1\le i,j\le n, i\neq j$ and $t_i=t_{a_0b_02}, 0\in\{a_0,b_0\}$, $t_j=t_{a_1b_12}$, $1 \in \{a_1, b_1\}$. From this it follows that

$f(2^{i-1},\alpha,2^{n-i})=0$, $f(2^{j-1}, \beta,2^{n-j})=1$, where $\alpha=0$ for $a_0=0$ and $\alpha=1$ for $b_0=1$; $\beta=0$ for $a_1=1$ and $\beta=1$ for $b_1=1$. In all cases $f \notin B_2$.

DEFINITION: Let $t^{(1)}=t_{002}, t^{(2)}=t_{012}, t^{(3)}=t_{022}$, $t^{(4)}=t_{112}$, $t^{(5)}=t_{122}$. A function f, which satisfies condition 2°, is of the form $(\alpha_1,\dots,\alpha_5), \alpha_i \in \{+, -\}$, $(1\le i\le 5)$, where $\alpha_i=+$ if and only if $t^{(i)} \in \{t_1,\dots,t_n\}$.

For example, the function $f=\min(t_{012}, t_{022}, t_{122})$ is of the form (-,+,+,-,+).

THEOREM 4. *All functions of the given form are in the same class of functions.*

P r o o f: Using Lemmas 3-11 we can conclude that a function, which satisfies conditions 2°, does not change the class if it is completed with some t_{abc} which is in the set $\{t_1,\dots,t_n\}(n\ge 2)$. In other words, if $\{t_1',\dots,t_{n'}'\}=\{t_1'',\dots,t_{n''}''\}$ and $n'\ge 2$, $n''\ge 2$, the functions $\min(t_1',\dots,t_{n'}')$ and $\min(t_1'',\dots,t_{n''}'')$ are of the same classes.

In Table 1 for every form $(\alpha_1,\dots,\alpha_5)$ the corresponding class is found. Since $n\ge 2$, the form (-,-,-,-,-) is excluded.

Using the given table and the consideration from 1°, we can easily conclude that the set $M_{min}\bar{M}_{max}$ contains the functions of 18 classes: *2, *5, *6, *9, *14, *17, *18, *19, *23, *28, *30, *31, *36, *37, *38, *39, *40, *44. For each of these classes, using Table 1, we can find the example of the function of this class. For example, $\min(t_{002},t_{122})$ is the function of class 2.

5. THE CLASSIFICATION OF THE SET $\bar{M}_{min}M_{max}$

Interchanging 0 and 2 in the definition of all the maximal sets T_0, $T_2, T_{01}, T_{12}, T_{02}, B_0, B_1, B_2, U_0$,U_2 ,T, M_{min}, M_{max} is mapped on the sets $T_2, T_0, T_{12}, T_{01}, T_{02}, B_2, B_1, B_0, U_2$,U_0 ,T, M_{max}, M_{min}. The set M_1 is mapped into M_1. For each of the classes *1-*48 in the table at the end of the paper, a similar class is given (the class obtained by interchanging 0 and 2). The classes of the set $\bar{M}_{min}M_{max}$ are similar to the classes of the set M_{min} $\bar{M}_{max}$.

These are: *1, *6, *5, *8, *16, *15, *19, *18, *25, *28, *29, *32, *35, *39, *40, *37, *38, *42.

Table 1.

N_0	t_{002}	t_{012}	t_{022}	t_{112}	t_{122}	T_0	T_2	T_{01}	T_{12}	T_{02}	B_0	B_1	B_2	U_0	U_2	T	No
1	+	+	+	+	+	0	0	0	1	1	1	1	1	1	1	1	2
2	+	+	+	+	-	0	0	0	1	1	1	1	1	1	1	1	2
3	+	+	+	-	+	0	0	0	1	1	1	1	1	1	1	1	2
4	+	+	+	-	-	0	0	0	1	0	0	1	1	1	1	1	9
5	+	+	-	+	+	0	0	0	1	1	1	1	1	1	1	1	2
6	+	+	-	+	-	0	0	0	1	1	1	1	1	1	0	1	6
7	+	+	-	-	+	0	0	0	1	1	1	1	1	1	1	1	2
8	+	+	-	-	-	0	0	0	1	0	0	1	1	1	0	1	17
9	+	-	+	+	+	0	0	0	1	1	1	1	1	1	1	1	2
10	+	-	+	+	-	0	0	0	1	1	1	1	1	1	1	1	2
11	+	-	+	-	+	0	0	0	1	1	1	1	1	1	1	1	2
12	+	-	+	-	-	0	0	0	1	0	0	1	0	1	1	0	30
13	+	-	-	+	+	0	0	0	1	1	1	1	1	1	1	1	2
14	+	-	-	+	-	0	0	0	1	1	1	1	1	1	0	1	6
15	+	-	-	-	+	0	0	0	1	1	1	1	1	1	1	1	2
16	+	-	-	-	-	0	0	0	1	0	0	1	0	1	0	0	37
17	-	+	+	+	+	0	0	0	0	1	1	1	1	0	1	1	14
18	-	+	+	+	-	0	0	0	0	1	1	1	1	0	1	1	14

19	-	+	+	-	+	0	0	0	0	1	1	1	1	0	1	1	14
20	-	+	+	-	-	0	0	0	0	0	0	1	1	0	1	1	36
21	-	+	-	+	+	0	0	0	0	1	1	0	1	0	1	1	23
22	-	+	-	+	-	0	0	0	0	1	1	0	1	0	0	1	28
23	-	+	-	-	+	0	0	0	0	1	1	0	1	0	1	1	23
24	-	+	-	-	-	0	0	0	0	0	0	0	1	0	0	1	44
25	-	-	+	+	+	0	0	0	0	1	1	1	1	0	1	1	14
26	-	-	+	+	-	0	0	0	0	1	1	1	1	0	1	1	14
27	-	-	+	-	+	0	0	1	0	1	1	1	1	0	1	1	5
28	-	-	+	-	-	0	0	1	0	0	0	1	0	0	1	0	39
29	-	-	-	+	+	1	0	0	0	1	1	0	0	0	1	0	31
30	-	-	-	+	-	1	0	0	0	1	1	0	0	0	0	0	38
31	-	-	-	-	+	1	0	1	0	1	1	0	0	0	1	0	18

6. THE CLASSIFICATION OF THE SET $\overline{M}_{min}\overline{M}_{max}$

From the above consideration it follows that the classes *3, *4, *7, *10, *11, *12, *13, *20, *21, *22, *24, *26, *27, *33, *34, *41, *43 do not contain the functions of the sets $M_{min}M_{max}$, $M_{min}\overline{M}_{max}$ and $\overline{M}_{min}M_{max}$. This implies that 17 classes from [4] are in the set $\overline{M}_{min}\overline{M}_{max}$. For the remaining 27 classes it is enough to investigate only 14 nonsimilar classes.

THEOREM 5. $M_1T_{02}U_0\ U_2\ B_0 \subseteq M_{min}$.

P r o o f: We shall prove that every nondegenerated function f of the set $M_1T_{02}U_0\ U_2\ B_0$ is of the form $f(x_1,\dots,x_n)=$ $=\min(x_1,\dots,x_n)$, and so f is in the set M_{min}.

Suppose that $f\in M_1T_{02}U_{12}U_{01}B_0$ and $f(x_1,\dots,x_n)\neq$ $\neq \min(x_1,\dots,x_n)$. Then one of the conditions 1° or 2° is satisfied.

1° There exist $(\alpha_1,\dots,\alpha_n)$ so that $0\in\{\alpha_1,\dots,\alpha_n\}$ and $f(\alpha_1,\dots,\alpha_n)\neq 0$. For $(\beta_1,\dots,\beta_n)$ such that $\beta_i=\alpha_i$ for $\alpha_i\neq 1$ and $\beta_i=2$ for $\alpha_i=1$ $(1\leq i\leq n)$ holds, since $f\in U_0\ T_{02}$, $f(\beta_1,\dots,\beta_n)=2$, $0\in\{\beta_1,\dots,\beta_n\}$.

2° There exists $(\alpha_1,\dots,\alpha_n)$ such that $1\in\{\alpha_1,\dots,\alpha_n\}$ and $f(\alpha_1,\dots,\alpha_n)=2$.

For $(\beta_1,\dots,\beta_n)$ such that $\beta_i=\alpha_i$ for $\alpha_i\neq 1$ and $\beta_i=0$ for $\alpha_i=1$ $(1\leq i\leq n)$ holds, since $f\in U_2\ T_{02}$, $f(\beta_1,\dots,\beta_n)=2$, $0\in\{\beta_1,\dots,\beta_n\}$.

So, in both cases, there exists $(\beta_1,\dots,\beta_n)$ such that $\beta_1,\dots,\beta_n=\{0,2\}$ and $f(\beta_1,\dots,\beta_n)=2$. Among all such n-sequences, let us choose such for which the number of 2 in n-sequences is the least possible. Let k (k<n) be the least such number of 2. Because of the similarity of the consideration suppose that such an n-sequence is $(2^k,0^{n-k})$ (Without loss of generality we can realize the permutation).

We shall prove that $f(x_1,\dots,x_n)=\min(x_1,\dots,x_k)$. We shall consider the following cases of possible n-sequences $(\alpha_1,\dots,\alpha_n)$.

1° $0\in\{\alpha_1,\dots,\alpha_k\}$.

We have to prove that for such n-sequences $f(\alpha_1,\dots,\alpha_n)=0$. We shall consider such n-sequences $(\beta_1,\dots,\beta_n)$ for which $\beta_i=\alpha_i$ for $\alpha_i\neq 1$ and $\beta_i = 2$ for $\alpha_i = 1$ $(1\leq i\leq n)$. Since $f\in T_{02}$ we have $f(\beta_1,\dots,\beta_n)\neq 1$. Suppose that $f(\beta_1,\dots,\beta_n)=2$. Let $(\gamma_1,\dots,\gamma_n)$ be such n-sequences that $\gamma_i=\beta_i$ for $1\leq i\leq k$. For $k+1\leq i\leq n$ let $\gamma_i=\beta_i$ for $\beta_i\neq 2$ and $\gamma_i=1$ for $\beta_i=2$. Since $f\in U_0$ we have $f(\gamma_1,\dots,\gamma_n)\neq 0$. The n-sequence $(\delta_1,\dots,\delta_n)$ ($\delta_i=0$ for $\gamma_i=1$, $\delta_i=\gamma_i$ for $\gamma_i\neq 1$) has less than k number of 2, and so $f(\delta_1,\dots,\delta_n)\neq 2$. Since $f\in U_2$ we have that $f(\gamma_1,\dots,\gamma_n)\neq 2$ and so $f(\gamma_1,\dots,\gamma_n)=1$.
The n-sequences $(\delta_1,\dots,\delta_n)$ are such that $\delta_i=2$ for $\gamma_i=0$ and $1\leq i\leq k$, and in the other cases is $\delta_i=\gamma_i$. Since $(2^k,0^{n-k})<(\delta_1,\dots,\delta_n)$ and $f\in M_1$ we have $f(\delta_1,\dots,\delta_n)=2$. Using $f(\gamma_1,\dots,\gamma_n)=1$ we obtain that $f\notin B_0$, which is a contradiction. So we conclude that $f(\beta_1,\dots,\beta_n)=0$. Since $f\in U_0$ it follows $f(\alpha_1,\dots,\alpha_n)=0$ and the proof is finished.

2^o $0 \notin \{\alpha_1,\ldots,\alpha_k\}$, $1 \in \{\alpha_1,\ldots,\alpha_k\}$.

We have to prove that $f(\alpha_1,\ldots,\alpha_n)=1$. Let us consider the n-sequences $(\beta_1,\ldots,\beta_n)$ such that $\beta_i=0$ for $k+1\leq i\leq n$, and the other i , $\beta_i=\alpha_i$. Since $f\in U_0$ and $f(2^k,0^{n-k})=2$ we have $f(\beta_1,\ldots,\beta_n)\neq 0$. The number of 2 in the n-sequence $(\beta_1,\ldots,\beta_n)$ is less than k and so $f(\beta_1,\ldots,\beta_n)\neq 2$ (we use also the property that $f\in U_2$). This implies that $f(\beta_1,\ldots,\beta_n)=1$. Since $(\alpha_1,\ldots,\alpha_k)\geq(\beta_1,\ldots,\beta_k)$, we have $f(\alpha_1,\ldots,\alpha_n)\neq 0$. Let us suppose that $f(\alpha_1,\ldots,\alpha_k)=2$ and consider such on n-sequence $(\beta_1,\ldots,\beta_n)$ where $\beta_i=2$ for $\alpha_i=2$ and $\beta_i=0$ for $\alpha_i\neq 2$ $(1\leq i\leq n)$. Since $f\in U_2$ we have $f(\beta_1,\ldots,\beta_n)=2$. But from $0\in\{\beta_1,\ldots,\beta_k\}$ we conclude that this is impossible, because of what was just proved in 1^o.

So we conclude that $f(\alpha_1,\ldots,\alpha_n)=1$ and the proof is finished.

3^o $\alpha_1=\ldots=\alpha_k=2$

Since $(\alpha_1,\ldots,\alpha_n)\geq(2^k,0^{n-k})$ and $f\in M_1$ it follows that $f(\alpha_1,\ldots,\alpha_n)=2$.

So the theorem is proved.

COROLLARY 1. The class *44(because of the similarity also the class *42) does not contain the functions of the set $\bar{M}_{min}\bar{M}_{max}$.

For the other 25 classes we shall give examples of functions of the set $\bar{M}_{min}\bar{M}_{max}$. The examples of the classes *2, *5, *9, *17, *23, *28, *30, *31 are also the examples of the corresponding classes from [4] . In order to check that the given examples and examples of the classes *14, *18, *36, *37, *38 are from the sets M_{min} and M_{max} we can use Table 1 and the similarity of the classes. Because of the similarity for the other 12 classes the examples are not given:

	0	1	2
0	0	0	1
1	0	0	1
2	0	2	2

*2

	0	1	2
0	0	1	1
1	1	2	2
2	1	2	2

*5

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

*9

	0	1	2
0	0	0	0
1	1	1	2
2	1	2	2

*14

	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

*17

	0	1	2
0	1	1	2
1	1	2	2
2	2	2	2

*18

	0	1	2
0	0	1	1
1	1	1	2
2	1	2	2

*23

	0	1	2
0	0	1	1
1	1	1	1
2	1	1	2

*28

	0	1	2
0	0	0	2
1	0	0	2
2	0	2	2

*30

	0	1	2
0	1	1	1
1	1	1	2
2	1	2	2

*31

	00	01	02	10	11	12	20	21	22
0	0	0	0	0	1	2	0	1	2
1	0	1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0	1	2

*36

	00	01	02	10	11	12	20	21	22
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	2	2	2

*37

	00	01	02	10	11	12	20	21	22
0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	2	2	2

*38

7. M_1-BASES

The set M_1 contains 88 classes of functions. These classes are cited at the end of the paper. By the algorithm given in [7] the numbers of M_1-bases and M_1-pivotal noncomplete sets are found. The obtained data are in the following table:

rank	1	2	3	4	5	6	7
Number of M_1-bases	0	0	1514	40104	75209	1916	1
Number of M_1-pivotal noncomplete sets	87	3153	37946	96323	15087	55	0

COROLLARY 2. The number of M_1-bases is 118744.

COROLLARY 3. The number of M_1-pivotal noncomplete sets is 152651.

COROLLARY 4. The maximal rank of the M_1-base is 7,and the minimal rank is 3.

COROLLARY 5. The maximal rank of M_1-pivotal noncomplete sets is 6.

COROLLARY 6. The unique M_1-base of the rank 7 is composed of the functions of the following classes: 81, 82, 83, 84, 85, 86, 87.

8. CLASSIFICATION OF M_1

NO	*NO	MIY	IS	T_0,	T_2	T_{01}	T_{12}	T_{02}	B_0	B_1	B_2	U_0	U_2	T	M_{min}	M_{max}	SIM
1	* 1	133	133	0	0	1	0	1	1	1	1	1	1	1	1	1	SIM *2
2	* 1	133	133	0	0	1	0	1	1	1	1	1	1	1	1	0	
3	* 2	134	134	0	0	0	1	1	1	1	1	1	1	1	1	1	SIM *1
4	* 2	134	134	0	0	0	1	1	1	1	1	1	1	1	0	1	
5	* 3	183	183	0	0	1	0	0	1	1	1	1	1	1	1	1	SIM *4
6	* 4	184	184	0	0	0	1	0	1	1	1	1	1	1	1	1	SIM *3
7	* 5	185	185	0	0	1	0	1	1	1	1	0	1	1	1	1	SIM *6
8	* 5	185	185	0	0	1	0	1	1	1	1	0	1	1	1	0	
9	* 5	185	185	0	0	1	0	1	1	1	1	0	1	1	0	1	
10	* 6	186	186	0	0	0	1	1	1	1	1	1	0	1	1	1	SIM *5
11	* 6	186	186	0	0	0	1	1	1	1	1	1	0	1	1	0	
12	* 6	186	186	0	0	0	1	1	1	1	1	1	0	1	0	1	
13	* 7	235	232	0	0	0	0	1	1	1	1	1	1	1	1	1	
14	* 8	236	233	0	0	1	0	0	1	1	0	1	1	1	1	1	SIM *9
15	* 8	236	233	0	0	1	0	0	1	1	0	1	1	1	1	0	
16	* 9	237	234	0	0	0	1	0	0	1	1	1	1	1	1	1	SIM *8
17	* 9	237	234	0	0	0	1	0	0	1	1	1	1	1	0	1	
18	* 10	238	235	0	0	1	0	0	1	1	1	0	1	1	1	1	SIM *11
19	* 11	239	236	0	0	0	1	0	1	1	1	1	0	1	1	1	SIM *10
20	* 12	291	282	0	0	0	0	0	1	1	1	1	1	1	1	1	
21	* 13	292	283	0	0	0	0	1	1	0	1	1	1	1	1	1	
22	* 14	293	284	0	0	0	0	1	1	1	1	0	1	1	1	1	SIM *16
23	* 14	293	284	0	0	0	0	1	1	1	1	0	1	1	0	1	
24	* 15	294	285	0	0	1	0	0	1	1	0	0	1	1	1	1	SIM *17
25	* 15	294	285	0	0	1	0	0	1	1	0	0	1	1	1	0	
26	* 16	295	286	0	0	0	0	1	1	1	1	1	0	1	1	1	SIM *14
27	* 16	295	286	0	0	0	0	1	1	1	1	1	0	1	1	0	
28	* 17	296	287	0	0	0	1	0	0	1	1	1	0	1	1	1	SIM *15
29	* 17	296	287	0	0	0	1	0	0	1	1	1	0	1	0	1	
30	* 18	320	308	1	0	1	0	1	1	0	0	0	1	0	1	1	SIM *19
31	* 18	320	308	1	0	1	0	1	1	0	0	0	1	0	1	0	
32	* 18	320	308	1	0	1	0	1	1	0	0	0	1	0	0	1	

NO	*NO	MIY	IS	T_0	T_2	T_{01}	T_{12}	T_{02}	B_0	B_1	B_2	U_0	U_2	T	M_{min}	M_{max}	SIM
33	* 18	320	308	1	0	1	0	1	1	0	0	0	1	0	0	0	
34	* 19	321	309	0	1	0	1	1	0	0	1	1	0	0	1	1	SIM *18
35	* 19	321	309	0	1	0	1	1	0	0	1	1	0	0	1	0	
36	* 19	321	309	0	1	0	1	1	0	0	1	1	0	0	0	1	
37	* 19	321	309	0	1	0	1	1	0	0	1	1	0	0	0	0	
38	* 20	346	334	0	0	0	0	0	1	1	0	1	1	1	1	1	SIM *21
39	* 21	347	335	0	0	0	0	0	0	1	1	1	1	1	1	1	SIM *20
40	* 22	348	336	0	0	0	0	0	1	1	1	0	1	1	1	1	SIM *24
41	*23	349	337	0	0	0	0	1	1	0	1	0	1	1	1	1	SIM *25
42	*23	349	337	0	0	0	0	1	1	0	1	0	1	1	0	1	
43	*24	350	339	0	0	0	0	0	1	1	1	1	0	1	1	1	SIM *22
44	*25	351	339	0	0	0	0	1	1	0	1	1	0	1	1	1	SIM *23
45	*25	351	339	0	0	0	0	1	1	0	1	1	0	1	1	0	
46	*26	373	361	0	0	0	0	0	1	1	0	0	1	1	1	1	SIM *27
47	*27	374	362	0	0	0	0	0	0	1	1	1	0	1	1	1	SIM *26
48	*28	375	363	0	0	0	0	1	1	0	1	0	0	1	1	1	
49	*28	375	363	0	0	0	0	1	1	0	1	0	0	1	1	0	
50	*28	375	363	0	0	0	0	1	1	0	1	0	0	1	0	1	
51	*29	390	378	0	0	1	0	0	0	1	0	1	1	0	1	1	SIM *30
52	*29	390	378	0	0	1	0	0	0	1	0	1	1	0	1	0	
53	*30	391	379	0	0	0	1	0	0	1	0	1	1	0	1	1	SIM *29
54	*30	391	379	0	0	0	1	0	0	1	0	1	1	0	0	1	
55	*31	392	380	1	0	0	0	1	1	0	0	0	1	0	1	1	SIM *32
56	*31	392	380	1	0	0	0	1	1	0	0	0	1	0	0	1	
57	*32	393	381	0	1	0	0	1	0	0	1	1	0	0	1	1	SIM *31
58	*32	393	381	0	1	0	0	1	0	0	1	1	0	0	1	0	
59	*33	399	387	0	0	0	0	0	0	1	0	1	0	1	1	1	
60	*34	400	388	0	0	0	0	0	1	0	1	0	1	1	1	1	
61	*35	401	389	0	0	0	0	0	1	1	0	1	0	1	1	1	SIM *36
62	*35	401	389	0	0	0	0	0	1	1	0	1	0	1	1	0	
63	*36	402	390	0	0	0	0	0	0	1	1	0	1	1	1	1	SIM *35
64	*36	402	390	0	0	0	0	0	0	1	1	0	1	1	0	1	
65	*37	405	393	0	0	0	1	0	0	1	0	1	0	0	1	1	SIM *39
66	*37	405	393	0	0	0	1	0	0	1	0	1	0	0	1	0	
67	*37	405	393	0	0	0	1	0	0	1	0	1	0	0	0	1	
68	*37	405	393	0	0	0	1	0	0	1	0	1	0	0	0	0	
69	*38	406	394	1	0	0	0	1	1	0	0	0	0	0	1	1	SIM *40
70	*38	406	394	1	0	0	0	1	1	0	0	0	0	0	1	0	
71	*38	406	394	1	0	0	0	1	1	0	0	0	0	0	0	1	
72	*38	406	394	1	0	0	0	1	1	0	0	0	0	0	0	0	
73	*39	407	395	0	0	1	0	0	0	1	0	0	1	0	1	1	SIM *37
74	*39	407	395	0	0	1	0	0	0	1	0	0	1	0	1	0	
75	*39	407	395	0	0	1	0	0	0	1	0	0	1	0	0	1	
76	*39	407	395	0	0	1	0	0	0	1	0	0	1	0	0	0	
77	*40	408	396	0	1	0	0	1	0	0	1	0	0	0	1	1	SIM *38
78	*40	408	396	0	1	0	0	1	0	0	1	0	0	0	1	0	
79	*40	408	396	0	1	0	0	1	0	0	1	0	0	0	0	1	
80	*40	408	396	0	1	0	0	1	0	0	1	0	0	0	0	0	
81	*41	411	399	0	0	0	0	0	0	1	0	1	0	1	1	1	SIM *43

82	*42	412	400	0	0	0	0	0	1	0	0	0	0	1	1	0	SIM *44
83	*43	413	401	0	0	0	0	0	0	1	0	0	1	1	1	1	SIM *41
84	*44	414	402	0	0	0	0	0	0	0	1	0	0	1	0	1	SIM *42
85	*45	415	403	1	0	1	0	0	0	0	0	0	0	0	0	0	SIM *47
86	*46	416	404	1	1	0	0	1	0	0	0	0	0	0	0	0	
87	*47	417	405	0	1	0	1	0	0	0	0	0	0	0	0	0	SIM *45
88	*48	418	406	0	0	0	0	0	0	0	0	0	0	0	0	0	

REFERENCES

[1] Яблонский С.В., Функциональные построения в к-значных логиках, Труды МИ АН СССР, Т 51, Москва, 1958, 5-142.

[2] Lau D., *Submaximale Klassen von P_3, Electronische Informationsverarbeitung und Kybernetik EIK 18(1982) 4/5, 227-243.*

[3] Machida H., *On Closed Set of Three-valued Monotone Logical Functions, Colloquia Mathematica Societatis* ***Janos Bolyai 28, North-*** *-Holland, 1981, 441-467.*

[4] Miyakawa M., *Functional Completeness and Structure of Three-valued Logic I-Classification of P_3, Researches of Electrotechnical Laboratory No. 717, 1971, 1-85.*

[5] Miyakawa M., *Enumeration of Bases of Three-valued Logical Functions, Colloquia Mathematica Societatis Janos Bolyai, 28, Szeged, North-Holland, 1981, 469-487.*

[6] Miyakawa M,. *Enumeration of Bases of a Submaximal Set of Three-valued Logical Functions, Rostocker Mathematisches Kolloquium, Heft 19, Wilhelm-Pieck-Universität Rostock, 1982, 49-67.*

[7] Stojmenović I.,*Classification of P_3 and Enumeration of the Bases of P_3,* ***Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, Ser.Mat., 14,1 1984, 73-80.***

[8] Stojmenović I., *Simetrične funkcije dvoznačne i troznačne logike, Ma-* ***ster's thesis, Novi Sad, 1983.***

Received by the editors September 6, 1983.

PREBROJAVANJE BAZA MONOTONIH FUNKCIJA TROZNAČNE LOGIKE

REZIME

U radu je odredjen broj tipova funkcija, broj tipova baza i broj tipova pivotalnih nekompletnih skupova za skup monotonih funkcija troznačne logike.