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A CONSTRUCTION OF SPECIAL k-SEMINETS

Janez Ušan
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

k-seminets [1] are a generalization of k-nets [5-6]. In [2] we considered a class of finite k-seminets, and a connection of a subclass of it with Sperner affine spaces is established [3-4]. In this paper we describe a construction of such k-seminets, using $\bar{k}$-nets which are in bijection with affine planes.

k-seminets, described in [1], represent a generalition of k-nets [5-6]. To each k-net there corresponds an orthogonal system of quasigroups, and conversely. In [1] it is shown that to each k-seminet there corresponds a regular orthogonal system of regular partial quasigroups, and conversely. Special k-nets characterize finite affine planes [5-6]. In [2] a class of finite k-seminets is considered, and a connection of a subclass of it with Sperner affine spaces is established [3-4]. In this paper we describe a construction of such k-seminets using $\bar{k}$-nets which are in bijection with affine planes.

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DEFINITION 1. [1] *Let* T *be a nonempty set, and* L *a nonempty set of subsets of* T. *Let the sets* $L_1,\dots,L_k$, $k \in N\setminus\{1,2\}$ *partition the set* L. *We say that the elements of* T *are points, the elements of* L *are lines, and the sets* $L_1,\dots,L_k$ *we call classes of lines.* $(T,L_1,\dots,L_k)$ *is said to be a k-seminet iff the following conditions are satisfied:*

M1. *The intersection of any two lines from different classes* L_i, L_j, $i,j \in \{1,2,\dots,k\}$ *is a set of cardinality at most* 1[1].

M2. *Each point from* L *belongs to exactly one line from the class* L_i, $i \in \{1,2,\dots,k\}$.

From M2 it follows that any two lines from the same class are disjoint, and from M2 and $k \in N\setminus\{1,2\}$ it follows that $|T| \geq 3$.

If in Definition 1 we replaced M1 with

$\overline{\text{M1}}$. The intersection of any two lines from different classes L_i, L_j, $i,j \in \{1,2,\dots,k\}$ is a set of cardinality 1[2] then $(T,L_1,\dots,L_k)$ is a k-net.

In this paper (as in [2] also) we consider only finite k-seminets, i.e. the k-seminets $(T,L_1,\dots,L_k)$ for which T is a finite set.

DEFINITION 2. [1] *Let* $(T,L_1,\dots,L_k)$ *be a finite k-seminet. Then, we say that* $\mathrm{Max}\{|\ell| \mid \ell \in L_1 \cup \dots \cup L_k\}$ *is the* **T**-*order and* $\mathrm{Max}\{|L_i| \mid i \in \{1,\dots,k\}\}$ *is the* **L**-*order of k-seminet* $(T,L_1,\dots,L_k)$.

The following statement is valid.

STATEMENT 1. [1] T-*order* $\leq$ L-*order*

In [2] the class of finite seminets $(T,L_1,\dots,L_k)$

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1) *We also say: Two lines from different classes intersect in at most one point.*

2) *We also say: Two lines from different classes intersect in exactly one point.*

is considered for which it holds:

M3. $(\forall \ell \in L_1 \cup \ldots \cup L_k) |\ell| = m \in N$ [1].

If for a k-seminet $(T, L_1, \ldots, L_k)$ M3 is satisfied, then the following condition is also satisfied:

M4. $|L_1| = \ldots = |L_k|$.

The converse is not always true.

STATEMENT 2. [2] *Let* $(T, L_1, \ldots, L_k)$ *be a* k-*seminet for which* L-*order* = q *and* T-*order* = m. *If for* $(T, L_1, \ldots, L_k)$ M3 *is satisfied, then*

(1) $$k \leq m \cdot \frac{q-1}{m-1} + 1$$ [2]

STATEMENT 3. [2] *Let for the* k-*seminet* $(T, L_1, \ldots, L_k)$ M3 *hold, and let* L-*order* = q *and* T-*order* = m. *Then*

(2) $$k = m \cdot \frac{q-1}{m-1} + 1$$

iff in $(T, L_1, \ldots, L_k)$ *any two points are colinear.*

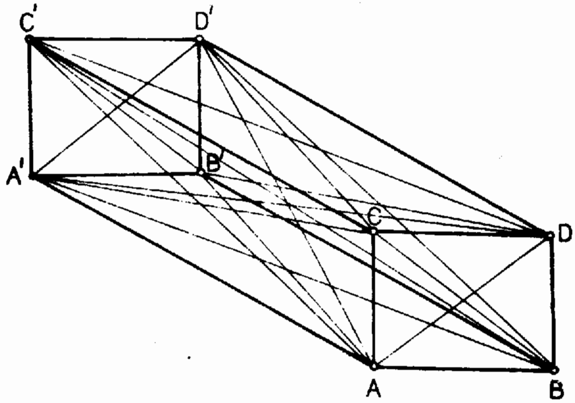

Fig. 1 [3]

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1) *Taking in mind M2 and* $k \in N \setminus \{1,2\}$, *we obtain that* $m \in N \setminus \{1\}$

2) *If* m=q, *then* (1) *become* $k \leq q+1$; *this represents the known property of* k-*nets* [5-6].

3) [2], *Fig.7.*

7-seminet from Fig. 1 is an example of a k-seminet satisfying M3 for which $m < q$ and equality (2) hold.

$$T = \{A,B,C,D,A',B',C',D'\} ,$$
$$L_1 = \{AB^{1)},CD,A'B',C'D'\} ,$$
$$L_2 = \{AC,BD,A'C',B'D'\} ,$$
$$L_3 = \{AA',BB',CC',DD'\} ,$$
$$L_4 = \{AD,BC,A'D',B'C'\} ,$$
$$L_5 = \{AC',A'C,BD',B'D\} ,$$
$$L_6 = \{AB',A'B,CD',C'D\} ,$$
$$L_7 = \{AD',A'D,CB',C'B\} .$$

The points of a 7-seminet are the vertices of a cube (Fig. 1). For the construction of this 7-seminet we used a 3-net of order 2, which is embedded in the faces and the diagonal planes of the cube. We generalize this construction by proving the following statement:

THEOREM 4. *Let* $(\overline{T},\overline{L}_1,\dots,\overline{L}_{\overline{k}})$ *be a* $\overline{k}$*-net of order* m *for which* $\overline{k} = m + 1$. *Then there exists a k-seminet* $(T,L_1,\dots,L_k)$ *with the property* M3 *for which* T-order = m, L-order = $q = m^2$ *and equality (2) holds.*

P r o o f. Let for the k-seminet $(T,L_1,\dots,L_k)$ the property M3 be satisfied, then $|L_1| = \dots = |L_k|$ = L-order; the converse is not always true [2]. A k-seminet with the property M3 for which T-order = m and L-order = $q = m^2$ has m^3 points.

Let the set T have m^2 points, $m \in N\setminus\{1\}$. Consider these points as the vertices of unit cubes stacked in an $m \times m \times m$ cube. We construct the classes $L_1,\dots,L_k$ in the following way:

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1) XY *is the abbreviation for* {X,Y}

1° The lines of a class are the sets of m points from T belonging to parallels to the chosen edge of the $m \times m \times m$ cubes. In this way we obtain three classes of lines. (In Figure 1, these classes are L_1, L_2 and L_3.).

2° In a face of the $m \times m \times m$ cube we embed the k-net $(\bar{T},\bar{L}_1,\dots,\bar{L}_{\bar{k}})$ of order m for which $\bar{k} = m + 1$, except the two classes described in 1°. The rest of this $\bar{k}$-net we translate to the opposite face, and to all the $m \times m$-layers of cube parallel to it. For two of the described lines, we say that they are corresponding iff one of them can be transformed into the other by using translation. Let, in the same class, there be all the lines of the same class of the rest of $\bar{k}$-net, lying in the face of the cube and all the lines corresponding to them. In this way $3(\bar{k} - 2) = 3(m - 1)$ new classes L_i are described.

3° Choose a face of the cube. For the set of all the points of corresponding lines, in the sense described in 2°, we say that they make a diagonal plane of the cube. Two diagonal planes are in the same class iff their "generating lines" from the $\bar{k}$-net $(\bar{T},\bar{L}_1,\dots,\bar{L}_{\bar{k}})$ embedded in a chosen face of the cube are in the same class $\bar{L}_i$[1)].

The number of different classes of diagonal planes, in a chosen face of the cube, is $\bar{k} - 2 = m - 1$, because two classes of the $\bar{k}$-net are already counted in the construction described in 1°.

In each diagonal plane we embed a $\bar{k}$-net from 2° in the following way (similarly as in 2°). Diagonal planes from the same class of diagonal planes (preserving the relation with the two classes of the $\bar{k}$-net described in 1°) first "develop" into the paralel $m \times m$ layers of the new $m \times m \times m$ cube (which is possible, because in the $\bar{k}$-net $(\bar{T},\bar{L}_1,\dots,\bar{L}_{\bar{k}})$ any two classes may be taken as the first

1) *In a class of diagonal planes there are* m *diagonal planes.*

two, and be embedded into the usual lines of the square grid $m \times m$); then we define the corresponding lines as in 2°. Now, the number of classes containing some of the new lines is $\bar{k}-2$; the lines of one class are described in 1°, and those of another class in 2°. In one class L_i let there be all the lines of the same class of the rest of the $\bar{k}$-net in the diagonal plane, and all the corresponding lines from the same class of diagonal planes. In this way we obtain the new $(\bar{k}-2)(\bar{k}-2) = (m-1)(m-1)$ classes L_i.

Analyzing the construction we can see that in the constructed system $(T,L_1,\ldots,L_k)$ M1, M2 and M3 are satisfied, and that $k = 3 + 3(m-1) + (m-1)(m-1) = m^2 + m + 1 = m \frac{q-1}{m-1} + 1$, which proves the statement.

Bearing in mind that Theorem 4 and the fact that the $\bar{k}$-net of order m satisfying the condition $\bar{k} = m+1$ exist if $m = p^{\alpha} \geq 3$, p prime, $\alpha \in N$ [4-6], we find that the following statement holds.

THEOREM 5. *If* $m = p^{\alpha} \geq 3$, p prime, $\alpha \in N$, $q = m^2$, *then there exists the* k-*seminet* $(T,L_1,\ldots,L_k)$ *with the property* M3, *with* L-order = q, T-order = m *and for which the equality (2) holds.*

In [2] [1)] a bijection is established between the k-seminets with the property M3, and for which the equality (2) holds, and the so called non-trivial Sperner spaces [3] [2)]. In that sense, Theorems 4 and 5 for non-trivial affine Sperner spaces hold too.

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1) *Theorems 6 and 7.*

2) [4], *pp. 293-294.*

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REZIME

JEDNA KONSTRUKCIJA SPECIJALNIH k-SEMIREŠETAKA

k-Semirešetke, koje je autor uveo u [1], predstavljaju jedno uopštenje k-rešetaka [5-6]. Svakoj k-rešetki odgovara ortogonalni sistem kvazigrupa, i obrnuto. U [1] je pokazano da svakoj k-semirešetki odgovara regularno ortogonalni sistem regularnih parcijalnih kvazigrupa, i obrnuto. Specijalne k-rešetke karakterišu konačne afine ravni [5-6]. U [2] je razmatrana jedna klasa konačnih k-semirešetaka i utvrdjena veza izmedju jedne njene podklase sa afinim prostorima Spernera [3-4]. U ovom radu se opisuje jedna konstrukcija takvih k-semirešetaka pomoću $\bar{k}$-rešetaka koje su u bijekciji sa afinim ravnima.