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FUZZY SETS ON S AS CLOSURE OPERATIONS ON P(S)

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ABSTRACT

It is known that a fuzzy set $\bar{A}$ on S ($S \neq \emptyset$), as a mapping from S onto a complete lattice L, determines a family $\{A_p; p \in L\}$ of subsets of S indexed by the elements of L (i.e. $\bar{A} = \bigcup_{p \in L} p \cdot A_p$, see for example [1]).

In [2] it is proved that the lattice of all L-valued fuzzy sets on S is isomorphic with the lattice of specially constructed mappings from L to $P(S)$. Using a similar method, we prove that the partially ordered set $\bar{A}_L = \langle\{A_p; p \in L\}, \subseteq\rangle$ is a complete lattice - a quotient relative to one closure operation in $\langle P(S), \subseteq\rangle$. The converse is also proved, i.e. that a special complete lattice $\bar{A}_L$ in $P(S)$, indexed by the elements of L, determines one fuzzy set $\bar{A}$ on S. Moreover, we prove that every fuzzy set $\bar{A} : S \to L$ determines a complete lattice $L_{\bar{A}}$, as a quotient of a closure in L, and that $L_{\bar{A}}$ is isomorphic with a dual lattice of $\bar{A}_L$. This proves that every fuzzy set $\bar{A} : S \to L$ is, up to the lattice isomorphism, equal to one fuzzy set $\breve{A} : S \to P(S)$.

We apply these results to the construction of a fuzzy

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congruence relation $\overline{\theta}$ on an algebra $\mathcal{A}$ ([3]), using the lattice of all congruence relations on $\mathcal{A}$.

<u>0.</u> Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a complete lattice, and $\overline{A} : S \to L$ a fuzzy set on S. Then

<u>0.1.</u> ([1]) . $\overline{A} = \bigcup_{p\in L} p\cdot A_p$. Here we use the following:

a) For every $p \in L$, $A_p \subseteq S$, and $x \in A_p$ iff $\overline{A}(x) \geq p$ $(x \in S)$;

b) We identify A_p with its characteristic function $A_p : S \to L$: if $x \in S$, then

$$A_p(x) = \begin{cases} 1, & \text{if } x \in A_p ; \\ 0, & \text{otherwise} ; \end{cases}$$

c) If $p \in L$, $x \in S$, then

$$p\cdot A_p(x) \overset{\text{def}}{=} p \wedge A_p(x) .$$

d) Using c), we get the mapping

$\overline{A}_p : S \to L$, $\overline{A}_p = p\cdot A_p$, i.e. for $x \in S$,

$$\overline{A}_p(x) = \begin{cases} p, & \text{if } x \in A_p , \\ 0, & \text{otherwise} ; \end{cases}$$

e) The union of the family of fuzzy sets defined in d), i.e. $\bigcup_{p\in L} \overline{A}_p$ is the usual fuzzy union ([1]), that is, it is a mapping from S to L, such that for $x \in S$

$$(\bigcup_{p\in L} \overline{A}_p)(x) = \bigvee_{p\in L} \overline{A}_p(x) .$$

Thereby, if $x \in S$

$$\overline{A}(x) = \bigvee_{p\in L} p\cdot A_p(x) .$$

1. Let $\bar{A} : S \to L$ be an arbitrary fuzzy set on S. Let, also, $\bar{A}_L = \langle\{A_p; p\in L\}, \subseteq\rangle$ be a partially ordered set of subsets of S, determined by $\bar{A}$, as in 0.1. Then

1.1. The partially ordered set $\bar{A}_L$ has the following properties:

(1) $\bigcap_{p\in L_1} A_p = A_{\bigvee_{p\in L_1} p}$ (where L_1 is an arbitrary subset of L);

(2) $\bar{A}_L$ is a complete lattice in which the infimum is the intersection.

P r o o f.

(1) Let $x \in S$. Then, $x \in \bigcap_{p\in L_1} A_p$ iff for every $p \in L_1$ $\bar{A}(x) \geq p$ iff $\bar{A}(x) \geq \bigvee_{p\in L_1} p$ iff $x \in A_{\bigvee_{p\in L_1} p}$, since L is complete.

(2) Using (1), we get

(1') If for $p,q \in L$, $q \in L$, $p \leq q$, then $A_q \subseteq A_p$, since $p \leq q$ implies by (1) $A_p \cap A_q = A_{p\vee q} = A_q$. Now, from (1') it follows that for every $p \in L$,

$$0 \leq p \text{ implies } A_p \subseteq A_0 \text{ , i.e.}$$

$A_0 = S$ is the greatest element in $\bar{A}_L$. Hence, using (1), we get that $\bar{A}_L$ is a complete lattice.

The following statement is a direct consequence of the preceding one.

1.2. Define the mapping $A \to \hat{A}$ on $P(S)$, such that $\hat{A} = \bigcap_{A\subseteq A_p} A_p$. This mapping is a closure operation on $P(S)$, and the lattice $\bar{A}_L$ is a quotient relative to that closure in a Boolean lattice $\langle P(S), \subseteq\rangle$.

P r o o f. A straightforward application of 1.1 and the wellknown properties of closure operations on lattices (see [5]).

The converse of 1.1 is the following proposition.

1.3. Let $A_p : L \to P(S)$ (for $p \in L$, $A_p \subseteq S$) be a mapping satisfying the following conditions:

(1) $\overline{A}_L = \langle \{A_p; p \in L\}, \subseteq \rangle$ is the complete lattice ;

(2) $\bigcap_{p \in L_1} A_p = A_{\bigvee_{p \in L_1} p}$ (for any subset L_1 of L);

(3) $A_0 = S$. (see also [2]).

Let $\overline{A} = \bigcup_{p \in L} p \cdot A_p$, where $p \cdot A_p$ is defined as in c), 0.1., and the union is defined as in e), 0.1.

Then $\overline{A}$ is a fuzzy set on S, and for $q \in L$

$$x \in A_q \quad \text{iff} \quad \overline{A}(x) \geq q .$$

P r o o f. $\overline{A}$ is clearly a fuzzy set on S, that is a mapping $S \to L$, since $p \cdot A_p$ is for every $p \in L$ a fuzzy set by construction, and the union of fuzzy sets is again a fuzzy set.

Now, let $x \in S$, $q \in L$, and $x \in A_q$. Then $\overline{A}(x) = \bigvee_{p \in L} p \cdot A_p(x) = \bigvee_{p \in L} \overline{A}_p(x) \geq q$, since $\overline{A}_q(x) = q$. Conversely, let $\overline{A}(x) \geq q$, and for $x \in S$, let $S_x = \{p; x \in A_p\}$. S_x is not empty, since by (3) $0 \in S_x$. Thus, $x \in \bigcap_{p \in S_x} A_p$. But

$$x \in \bigcap_{p \in S_x} A_p \quad \text{iff} \quad x \in A_{\bigvee_{p \in S_x} p} \quad \text{iff} \quad x \in A_{\bigvee S_x} .$$

We also have

$$q \leq \bigvee_{p \in L} p \cdot A_p(x) = \bigvee_{p \in L} \overline{A}_p(x) = \bigvee S_x, \quad \text{i.e.} \quad q \leq \bigvee S_x .$$

Using (2), we get $A_{\bigvee S_x} \leq A_q$, and since $x \in A_{\bigvee S_x}$, it follows that $x \in A_q$.

Proposition 1.3 has following two simple corollaries.

1.4. If $A_p = \emptyset$, and $p \leq q$, then $A_q = \emptyset$.

1.5. If $A_q \subseteq A_p$, then $A_q = A_{q \vee p}$.

2. Let $\bar{A} = \bigcup_{p \in L} p \cdot A_p$ be an arbitrary fuzzy set on S. Define a relation $\sim$ on L:

$$p \sim q \quad \text{iff} \quad A_p = A_q .$$

2.1. $\sim$ is an equivalence relation on L.

P r o o f. Straightforward.

2.2. Let $p, q \in L$. Then,

$$p \leq q \quad \text{implies} \quad q \vee p \sim p .$$

P r o o f. By 1.5.

Define now a mapping $p \to \bigvee_{q \in |p|_\sim} q$ on L, and let $\bigvee_{q \in |p|_\sim} q = p_m$.

2.3. $p_m \in |p|_\sim$.

P r o o f. If $q \in |p|_\sim$, then

$$A_p = A_q = \bigcap_{q \in |p|_\sim} A_q = A_{\bigvee_{q \in |p|_\sim} q} = A_{p_m}, \quad \text{i.e.} \quad p \sim p_m.$$

Note that 2.3 is equivalent with $|p_m|_\sim = |p|_\sim$.

2.4. The mapping $p \to p_m$ is a closure operation on L.

P r o o f. 1° Since $p \in |p|_\sim$, it follows that

$$p \leq \bigvee_{q \in |p|_\sim} q = p_m, \quad \text{i.e.} \quad p \leq p_m .$$

2° If $p \leq q$, then $A_{q_m} = A_q \leq A_p = A_{p_m}$,

i.e. $A_{q_m} \subseteq A_{p_m}$. Hence $q_m \vee p_m \sim q_m$ and $q_m \vee p_m \in |q_m|_\sim$. Thus

$$q_m \vee p_m \leq q_m, \quad \text{i.e.} \quad p_m \leq q_m.$$

3° From $|p|_{\sim} = |p_m|_{\sim}$, and $(p_m)_m = \bigvee_{q \in |p_m|_{\sim}} q = \bigvee_{q \in |p|_{\sim}} q = p_m$, it follows that $(p_m)_m = p_m$.

1°, 2° and 3° prove the proposition.

Considering the set of closed elements under the closure operation defined above, we get the following corollary.

2.5. The partially ordered set $L_{\bar{A}} = \langle \{p \in L;\ p = p_m\}, \leq \rangle$, where the ordering relation is the one from L, is a complete lattice - the quotient relative to the closure operation $p \to p_m$.

P r o o f. This is a general property of closure operations on lattices (see, for example [5]).

To discuss the connection between $\bar{A}_L$ and $L_{\bar{A}}$ we need the following lemma.

2.6. If $p, q \in L_{\bar{A}}$, and $p \sim q$, then $p = q$.

P r o o f. If $p \sim q$, then $p \leq q_m = q$, and $q \leq p_m = p$, i.e. $p = q$.

Let $\bar{A}_{dL} = \langle \{A_p; p \in L\}, \preccurlyeq \rangle$ be the dual lattice of $\bar{A}_L$. Clearly, $A_p \preccurlyeq A_q$ iff $A_q \subseteq A_p$.

2.7. $L_{\bar{A}} \cong \bar{A}_{dL}$.

P r o o f. Consider the mapping $f : \{p \in L;\ p = p_m\} \to \{A_p; p \in L\}$, such that $f(p) = A_p$.

(i) f is "onto": really, if $A_q \in \{A_p; p \in L\}$, then $A_q = A_{q_m}$, and $f(q_m) = A_{q_m} = A_q$.

(ii) f is "one to one": $f(p) = f(q)$ iff $A_p = A_q$ iff $p \sim q$ iff (by 2.6.) $p = q$.

(iii) f preserves the ordering: if $p \leq q$, then $A_q \subseteq A_p$ i.e. $A_p \preccurlyeq A_q$.

(iv) f^{-1} preserves the ordering: let $A_p \preccurlyeq A_q$, i.e. $A_q \subseteq A_p$.

Then $A_{q_m} \subseteq A_{p_m}$, since $A_p = A_{p_m}$. Thereby $A_{q_m} = A_{p_m} \cap A_{q_m} = A_{p_m \vee q_m}$ i.e. $p_m \vee q_m \in |q_m|_\sim$, and thus $p_m \vee q_m \leq q_m$, i.e. $p_m \leq q_m$.

By (i), (ii), (iii) and (iv), f is a lattice isomorphism.

Let now for $p \in L_{\bar{A}}$ (determined by a given fuzzy set $\bar{A}$ on S), and for $x \in S$

$$p \cdot A_p(x) = \begin{cases} p, & \text{if } x \in A_p \\ 0 \in L_{\bar{A}}, & \text{otherwise} . \end{cases}$$

(Note that $|0|_\sim = 0$ (i.e. $0_{L_{\bar{A}}} = 0_L$), iff $\mathrm{card}(|0|_\sim) = 1$).

<u>2.8.</u> If $\bar{A} = \bigcup_{p \in L} p \cdot A_p$, then $\bar{A} = \bigcup_{p \in L_{\bar{A}}} p \cdot A_p$.

P r o o f. If $x \in S$, then

$$\bar{A}(x) = \bigvee_{p \in L} p \cdot A_p(x) = \bigvee_{|p|_\sim \in L/_\sim} \Big(\bigvee_{q \in |p|_\sim} q \cdot A_q(x) \Big) = \bigvee_{p \in L_{\bar{A}}} p \cdot A_p(x) .$$

The last supremum is the same in L and in $L_{\bar{A}}$, since it is a supremum of the elements of $L_{\bar{A}}$, and the supremum itself belongs to $L_{\bar{A}}$ (otherwise, if $\bar{A}(x) = p \notin L_{\bar{A}}$, then $p < p_m$ and $A_p = A_{p_m}$. Hence $x \in A_{p_m}$, and $\bar{A}(x) \geq p_m$, which is a contradiction, proving that $\bar{A}(x) \in L_{\bar{A}}$ for every $x \in S$).

We have thus proved that for every fuzzy set $\bar{A} : S \to L$, there is a lattice $L_{\bar{A}} \subseteq L$ (being a quotient of a closure in L), such that $\bar{A}$ is a mapping from S to $L_{\bar{A}}$, i.e. that 2.8 . holds. Moreover, using the isomorphism from $L_{\bar{A}}$ to $\bar{A}_{dL} = \langle S_1, \preccurlyeq \rangle$, where $S_1 = \{A_p ; p \in L\} \subseteq P(S)$, one can consider $\bar{A}$ as a mapping from S to P(S).

<u>2.9.</u> Let $\bar{A} : S \to L$ be an arbitrary fuzzy set on S. Also let f be the isomorphism from $L_{\bar{A}}$ to $\bar{A}_{dL}$, defined in 2.7, and let $\check{A} : S \to P(S)$ be the mapping for which $\check{A}(x) \overset{def}{=} f(\bar{A}(x))$. Then, if $x \in S$

$$\check{A}(x) = \bigcap_{\substack{P \in S_1 \\ x \in P}} P.$$

(Remark: If $\overline{A} = \bigcup_{p\in L_{\overline{A}}} p\cdot A_p$ and $\check{A} = \bigcup_{f(p)\in S_1} f(p)\cdot A_p$,[1)] then $f(\overline{A}(x)) = \check{A}(x))$.

P r o o f. $\overline{A} = \bigcup_{p\in L} p\cdot A_p = \bigcup_{p\in L_{\overline{A}}} p\cdot A_p$, by 2.8. Now, since $L_{\overline{A}} \cong \overline{A}_{dL}$, and since $f(p) = A_p$,

$$\check{A} = \bigcup_{p\in L_{\overline{A}}} f(p)\cdot A_p = \bigcup_{f(p)\in S_1} f(p)\cdot f(p).$$

Here $f(p)\cdot A_p = f(p)\cdot f(p)$ is a mapping $S \to P(S)$, such that

$$f(p)\cdot f(p)(x) = \begin{cases} f(p), & \text{if } x \in f(p) = A_p\ , \\ 0_{\overline{A}_{dL}} = S, & \text{otherwise,} \end{cases}$$

since $0_{\overline{A}_{dL}} = 1_{\overline{A}_L} = \overline{A}_o = S$. Hence for $x \in S$ $\check{A}(x) = \bigvee_{\substack{P\in S_1\\ x\in P}} P$, where the supremum is in $\overline{A}_{dL}$, the dual lattice of $\overline{A}_L$. Since the infimum in $\overline{A}_L$ is the set intersection, it follows that $\check{A}(x) = \bigcap_{\substack{P\in S_1\\ x\in P}} P$.

3. We shall now apply the results from 2 in describing the most general method of constructing the fuzzy congruence relations on an algebra $\mathcal{A}$ ([4]), using only the lattice of all (ordinary) congruence relations on $\mathcal{A}$. This proves that, even for such a special notion as a fuzzy congruence relation, the choice of a lattice $\mathcal{L}$ is equvalent with the choice of a closure operation in a lattice of all congruence relation on $\mathcal{A}$.

Let $\mathcal{A} = \langle S, \mathcal{O}\rangle$ be an algebra, and $\mathcal{L}$ an arbitrary complete lattice. The mapping $\overline{\theta} : S^2 \to L$ is a *fuzzy congruence relation* on $\mathcal{A}$, iff the following conditions are satisfied ([3]):

(r) For all $x \in S$, $\overline{\theta}(x,x) = 1 \in \mathcal{L}$;

(s) For all $x,y \in S$, $\overline{\theta}(x,y) = \overline{\theta}(y,x)$;

(t) For all $x,y \in S$, $\overline{\theta}(x,y) \geq \bigvee_{z\in S} (\overline{\theta}(x,z) \wedge \overline{\theta}(z,y))$;

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1) *In* $f(p)\cdot A_p$, A_p *is the characteristic function of* A_p *in the sence of 0.1.b), on the lettice* $\overline{A}_{dL}$.

(c) If $x_1,\dots,x_n,y_1,\dots,y_n \in S$, $f \in F(n) \subseteq O$, and $\overline{\theta}(x_i,y_i) = p_i$, $i = 1,\dots,n$, then
$$\overline{\theta}(f(x_1,\dots,x_n),f(y_1,\dots,y_n)) \geq \bigwedge_{i=1}^{n} p_i .$$

From 2.9 we know that every fuzzy set on S can be indexed by the elements of $P(S)$, and from 1.3, if we use $P(S)$ instead of L, we get the construction of fuzzy set using $P(S)$ only. It is clear that the same method is applicable on the construction of fuzzy relations. Moreover, we claime that the properties of fuzzy congruence relations ([3]) are preserved under the described constructions.

3.1. Let $A = \langle S,O\rangle$ be an algebra, and let $C = \langle C,\subseteq\rangle$ be a lattice of its congruence relation. Let C_m be a subset of C closed under the arbitrary intersections, and which contains S^2 and the diagonal ε. Then $\overline{\theta} : S^2 \to C_m$, such that $\overline{\theta}(x,y) = \bigcap_{(x,y)\in\sigma}\sigma$ is a fuzzy congruence relation on A.

Proof. $\overline{\theta}$ is a reflexive, since $\overline{\theta}(x,x) = \bigcap_{(x,x)\in\sigma}\sigma = \bigcap \zeta_m = \varepsilon$ (diagonal). $\overline{\theta}$ is obviously symmetric. It is transitive:
$\overline{\theta}(x,z) \wedge \overline{\theta}(z,y) = \bigcap_{(x,z)\in\sigma_i}\sigma_i . \wedge \bigcap_{(z,y)\in\tau_i}\tau_i = (\sigma \wedge \tau \mid (x,z) \in \sigma,$

$(z,y) \in\tau) = (\rho \mid (x,z) \in \rho,\ (z,y) \in \rho) = (\rho \mid (x,y) \in \rho)$ (ρ is transitive).

Thereby, $\overline{\theta}(x,y) = \bigcap_{(x,y)\in\rho_i}\rho_i \subseteq \rho = \overline{\theta}(x,z) \wedge \overline{\theta}(z,y)$.

Hence, $\overline{\theta}(x,y) \subseteq \bigcap_{z\in S}(\overline{\theta}(x,z) \wedge \overline{\theta}(z,y))$, i.e.

$$\overline{\theta}(x,y) \geqslant \bigvee_{z\in S}(\overline{\theta}(x,z) \wedge \overline{\theta}(z,y)) .$$

$\overline{\theta}$ satisfies the substitution properta (c): (we give the proof for a binary operation " $\cdot$ ", the same is in the general case); let

$$\overline{\theta}(x_1,y_1) = \sigma_1 = \bigcap_{(x_1,y_1)\in\sigma_i}\sigma_i ,$$

$$\overline{\theta}(x_2,y_2) = \sigma_2 = \bigcap_{(x_2,y_2)\in\tau_i} \tau_i. \quad \text{Then}$$

$_1 \wedge \sigma_2 = \tau$ and $(x_1,y_1) \in \tau$, $(x_2,y_2) \in \tau$. Since
$\in \zeta_m \subseteq C$, $(x_1\cdot x_2, y_1\cdot y_2) \in \tau$, and hence

$$\overline{\theta}(x_1\cdot x_2, y_1\cdot y_2) = \bigcap_{(x_1\cdot x_2,y_1\cdot y_2)\in\sigma} \sigma \subseteq \tau, \text{ proving (c).}$$

Note that by 2.9 every fuzzy congruence relation on is equal up to the lattice isomorphism, to the one consructed in 3.1, which proves that the lattice of all congruence relations on A is all we need in considering the fuzzy ɔngruence relations on A.

ꓤEFERENCES

1] Kaufman A., *Introduction a la theorie des sous-ensembles floues, Paris, 1973.*

2] Botta O., *Theoreme de representation des parties floues, Seminaire: Mathematique floue, 1978-79.*

3] Vojvodić G., Šešelja B., *O strukturi slabih relacija ekvivalencije i slabih relacija kongruencije, Matematički Vesnik, 1(14)(29) 1977, 147-152.*

4] Vojvodić G., Šešelja B., *On Fuzzy Quotient Algebras, Zb.rad.Prir.-mat.fak., Novi Sad, Ser.Mat., 13(1983), 279-288.*

5] Aigner M., *Combinatorial theory, Springer-Verlag, 1979.*

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:ZIME

RASPLINUTI SKUPOVI NA S KAO OPERATORI ZATVARANJA NA P(S)

U radu se pokazuje da svaki rasplinuti skup $S \neq \emptyset$:redjuje jednu kompletnu mrežu, količnik posebnog zatvorenja P(S), čiji su elementi indeksirani elementima odgovarajuće mreže L. Pokazuje se da postoji izomorfizam tako konstruisane mreže podskupova od S i količnika drugog zatvorenja - kompletne mreže sadržane u L. Tako se pokazuje da se svaki rasplinuti skup S (kao preslikavanje $S \to L$) može do na mrežni izomorfizam tretirati kao preslikavanje S u P(S). Ovi rezultati primenjeni su na konstrukciju rasplinutih kongruencija na algebri A, uz pomoć mreže svih kongruencija te algebre.