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ONE CONNECTION BETWEEN BINARY AND(n+1)-ARY EQUIVALENCE RELATIONS ON FINITE SETS

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ABSTRACT

In this article we shall prove that every (n+1)-ary equivalence relation on a set S uniquely determines one binary equivalence in the set $S^{(n)}$ of all n-subsets of S, satisfying a special property ((i_n), Lemma 2). We shall also prove the converse, i.e. that there is a bijection between these two sets of relations. One of its consequences is (Corallary 11) that every lattice of (n+1)-ary equivalences (or of partitions of type n) the finite set S is isomorphic to the quotient relative to one closure operation on the lattice of binary equivalences (or of partitions of type 1) on $S^{(n)}$.

Since every finite lattice is embeddable into the lattice of (n+1)-ary equivalences ([1]), this also proves that every finite lattice is embeddable into the lattice of binary equivalences, satisfying the property (i_n) (Lemma 2).

In [1] Hartimanis defined a partition of type n, and in [2] Pickett gave a suitable definition of a corresponding (n+1)-ary equivalence relation on the same set. Some characterizations of these equivalences were given in [4] . One class of (n+1)-ary quasiordering relations and a lattice of included (n+1)-ary equivalences wes considered in [5] (covering thus the contents of [3]), and the corresponding generalized orderings were discussed in [6]. Some operations on the set of all (n+1) ary relations on $S\neq\emptyset$, inculuding some closure operators, were defined in [8]. In [7], it wes proved that every (n+1) ary equivalence can be represented by the system of equivalences of any lower arity.

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1. An (n+1)-ary relation ρ on the set $S\neq\emptyset$ is (i,j)-reflexive, $i \neq j$, $i, j \in \{1,\dots,n+1\}$, iff

$$(\forall a_1,\dots,a_{j-1},a_{j+1},\dots a_{n+1} \in S)\ ((a_i^{i-1},a_i,a_{i+1}^{j-1},a_i,a_{j+1}^{n+1}) \in\rho)^{1)}.$$

ρ is reflexive iff it is (i,j)-reflexive for all $i,j\in\{1,\dots,n+1\}$, $i \neq j$ [2].

2. An (n+1)-ary relation ρ on S is π-symmetric, $\pi \in \{1,\dots,n+1\}!$ iff

$$(\forall a_1,\dots,a_{n+1} \in S)((a_1^{n+1}) \in \rho \Longrightarrow (a_{\pi(1)},\dots,a_{\pi(n+1)}) \in \rho).$$

([2]) ρ is symmetric iff it is π-symmetric for all $\pi \in \{1,\dots,n+1\}!$,

3. An (n+1)-ary relation ρ on S is $i\bar{A}_1$-transitive[3] $i \in \{1,\dots,n\}$, iff

$$(\forall a_0,\dots,a_{n+1} \in S)(a_0^{i-1},a_i a_{i+1}^{n}) \in \rho \wedge (a_1^{i-1},a_i a_{i+1}^{n+1}) \in \rho \wedge$$

$$(a_j\neq a_k,\ \text{for } j\neq k,\ j,k \in \{1,\dots,n\})\Longrightarrow (a_0^{i-1},a_{i+1}^{n+1}) \in \rho).$$

4_1. [1] For set S with at least n elements, the family P_n of subsets of S is a partition of type n, iff

(1) each member of P_n has at least n elements, and

(2) any n different elements from S belong to exactly one number of P_n.

4_2. [2] An (n+1)-ary relation ρ is a generalized equivalence relation of S iff it satisfies:

E1n: (1,n+1)-reflexivity,

E2n: symmetry, and

E3n: $n\bar{A}_1$-transitivity.

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1) a_p^q stands for $a_p,a_{p+1},\dots,a_{q-1}$, and it is empty when $q<p$; consequently, a_p^p is a_p, and instead of a,...,a(n times), we write $\overset{n}{a}$; $\overset{o}{a}$ is, clearly, empty.

2) In [2] an (n+1)-ary (1,n+1)-reflexive relation is called "reflexive".

3) In [2] : "transitive" stands for $n\bar{\bar{A}}_1$-transitive.

PROPOSITION 1. *Let S be a finite set,* $|S|>n$, $n\in N$, *and let* $S^{(n)}=\{\{a_1,\dots,a_n\}\,|\,a_1,\dots,a_n$ *are different elements from* $S\}$

If ρ *is an arbitrary* $(n+1)$*- ary equivalence relation on* S, *let us define a binary relation* $\rho_{(n)}$ *on* $S^{(n)}$: *If* $A,B\in S^{(n)}$

(1_n) $\quad (A,B)\in\rho_{(n)}$ *if* $(A\cup B)^{n+1}\subseteq\rho$.

$\rho_{(n)}$ *is a binary equivalence relation of* $S^{(n)}$.

P r o o f.

1°. $\rho_{(n)}$ is reflexive:

If $A\in S^{(n)}$ then $(A,A)\in\rho_{(n)}$ iff $A^{n+1}\in\rho$. $|A|=n$ and thus $(x_1^{n+1})\in A^{n+1}$ iff $x_i=x_j$ for some $i\neq j$, $i,j\in\{1,\dots,n+1\}$. ρ is (i,j)-reflexive for all $i,j\in\{1,\dots,n+1\}$ and thus $\rho_{(n)}$ is reflexive.

2° $\rho_{(n)}$ is symmetric:

If for $A,B\in S^{(n)}$, $(A,B)\in\rho_{(n)}$ then $(A\cup B)^{n+1}\subseteq\rho$ i.e. $(B\cup A)^{n+1}\subseteq\rho$, and thus $(B,A)\subset\rho_{(n)}$.

3° $\rho_{(n)}$ is transitive:

Let $A,B,C\in S^{(n)}$, $(A,B)\in\rho_{(n)}$ and $(B,C)\in\rho_{(n)}$. Then

(a) $\quad (A\cup B)^{n+1}\subseteq\rho$ and $(B\cup C)^{n+1}\subseteq\rho$.

Consider now an arbitrary $(n+1)$-tuple $(a_1^k,a_{k+1}^{n+1})\in(A\cup C)^{n+1}$, where $a_1,\dots,a_k\in A$, $c_{k+1},\dots,c_{n+1}\in C$, $k\in\{0,\dots,n+1\}$. If $a_1,\dots,a_k,c_{k+1},\dots,c_{n+1}$ are not all different, then (a_1^k,c_{k+1}^{n+1}) ρ, since ρ is (i,j)-reflexive for all i,j. If $a_1,\dots,a_k$, $c_{k+1},\dots,c_{n+1}$ are all different, we proceed in the following way:

Let $k=1$, and for all $\alpha,\beta\in\{1,\dots,n\}!$ let $a_{\alpha(1)}\stackrel{def}{=}\bar{a}_1$, $c_{\beta(1)}=\bar{c}_1,\dots,c_{\beta(n)}=\bar{c}_n$. Now it follows from (a) that

$$(\bar{a}_1,\bar{b}_1^n) \in \rho \quad \text{and} \quad (\bar{b}_1^n, \bar{c}_1) \in \rho ,$$

where $\bar{b}_1,\ldots,\bar{b}_n$ are different elements from B.

Since ρ is $n\bar{A}_1$-transitive and symmetric, we get $(\bar{a}_1, \bar{b}_1^{n-1}, \bar{c}_1) \in \rho$, that is,

(a1) $$(\bar{a}_1,\bar{c}_1\bar{b}_1^{n-1}) \in \rho .$$

From (a) we also have

(a2) $$(\bar{c}_1,\bar{b}_1^{n-1},\bar{c}_2) \in \rho .$$

The transitivity now from (a1) and (a2) gives $(\bar{a}_1,\bar{c}_1,\bar{b}_1^{n-2}, \bar{c}_2) \in \rho$, (we can assume here that $\bar{c}_1 \notin \{\bar{b}_2,\ldots,\bar{b}_{n-1}\}$, since $C \neq B$ and we choose $\bar{c}_1$) and hence by the symmetry of ρ

(a3) $$(\bar{a}_1,\bar{c}_1,\bar{c}_2,\bar{b}_1^{n-2}) \in \rho .$$

Now we can assume that $\bar{c}_1,\bar{c}_2,\bar{b}_1,\ldots,\bar{b}_{n-2}$ are all different (since $\bar{c}_2$ can be equal to at least one of the n different elements from B and we omit exactly this one, say $\bar{b}_{n-1}$). We now apply the transitivity on (a3) and on

(a4) $$(\bar{c}_1,\bar{c}_2,\bar{b}_1^{n-2},\bar{c}_3) \in \rho \text{ (by assumption (a))}$$

and we get

$$(\bar{a}_1,\bar{c}_1,\bar{c}_2,\bar{b}_1^{n-3},\bar{c}_3) \in \rho .$$

It is now clear that this procedure in a finite number of steps) gives

(b1) $$(\bar{a}_1,\bar{c}_1^n) \in \rho .$$

But if we start with $\bar{a}_2$, we get

$$(\bar{a}_2,\bar{c}_1^n) \in \rho \quad \text{i.e.} \quad (\bar{c}_1^n,\bar{a}_2) \in \rho ,$$

which, with (b_1) implies $(\bar{a}_1,\bar{c}_1^{n-1},\bar{a}_2) \in \rho$.

The same application of transitivity, again in a finite number of steps, finally gives

$$(\bar{a}_1^k, \bar{c}_{k+1}^{n+1}) \in \rho,$$

proving by (1_n) that (n) is transitive. $1^0, 2^0$ and 3^0 prove the proposition completely.

LEMMA 2. *Let* $\rho, \rho_{(n)}$ *be as in Proposition 1., and* $A, B \in S^{(n)}$. *Then,*

(i_n) $(A,B) \in \rho_{(n)}$ *iff for all* $x, y, \in (A \cup B)^{(n)}$, $(x,y) \in \rho_{(n)}$.

P r o o f. If for all $X, Y \in (A \cup B)^{(n)}$, $(X,Y) \in \rho_{(n)}$, then it is clear that also $(A,B) \in \rho_{(n)}$.

If $(A,B) \in \rho_{(n)}$, and $X, Y \in (A \cup B)^{(n)}$, then

$X \cup Y \subseteq A \cup B$, $(X \cup Y)^{(n)} \subseteq (A \cup B)^{(n)} \subseteq \rho$, and thus by (1_n) $(X,Y) \in \rho_{(n)}$.

The following proposition describes the blocks of the partition (of type 1) determined by $\rho_{(n)}$.

COROLLARY 3. *If* $C_{\{x_1^n\}} \in S^{(n)}/\rho_{(n)}, \{x_1^n\} \in S^{(n)}$, *then*

$$C_{\{x_1^n\}} = (\cup C_{\{x_1^n\}})^{(n)\ 1)}.$$

P r o o f. Let $\{y_1^n\} \in (\cup C_{\{x_1^n\}})^{(n)}$. Then, $y_1 \in A_1, \ldots$ $\ldots, y_n \in A_n$ for some $A_1, \ldots, A_n \in C_{\{x_1^n\}}$. From $(A_1, A_2) \in \rho_{(n)}$, by Lemma 2., it follows that for some $A_1' \in (A_1 \cup A_2)^{(n)}$, $(A_1', A_2) \in \rho_{(n)}$, with $y_1, y_2 \in A_1'$. Also, since $(A_2, A_3) \in \rho_{(n)}$, and since $\rho_{(n)}$ is transitive, it follows that $(A_1', A_3) \in \rho_{(n)}$, and there is $A_1'' \in (A_1' \cup A_3)^{(n)}$, such that $(A_1'', A_3) \in \rho_{(n)}$, and $y_1, y_2, y_3 \in A_1''$. Continuing this procedure, we get $(\{y_1^n\}, A_n) \in \rho_{(n)}$, i.e. $(\{y_1^n\}, \{x_1^n\}) \in \rho_{(n)}$, and hence $\{y_1^n\} \in C_{\{x_1^n\}}$.

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1) $\{x_1^n\}$ is the abbreviation of $\{x_1, \ldots, x_n\}$.

Thus, $(\bigcup C_{\{x_1^n\}})^{(n)} \subset C_{\{x_1^n\}}$. It is obvious that $C_{\{x_1^n\}} \subseteq (\bigcup C_{\{x_1^n\}})^{(n)}$, and thus the proof is complete.

The following two propositions prove that the connection between (n+1)-ary equivalences on S and binary equivalences on $S^{(n)}$ is in fact a bijection.

Proposition 4. *If* ρ *is anarbitrary* (n+1)*-ary equivalence relation on* S, *and* $\rho_{(n)}$ *is a binary equivalence on* $S^{(n)}$ *defined in Proposition 1 (with* (1_n)*), then*

$$(j_n) \qquad \rho = \bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1}$$

P r o o f . (i) Let $(x_1^{n+1}) \in \rho$. If $x_i = x_j$ for some $i \neq j$, $i, j \in \{1,\dots,n+1\}$, then $|\{x_1^{n+1}\}| \leq n$, and let $X \in S^{(n)}$, $\{x_1^{n+1}\} \subset X$. $\rho_{(n)}$ is reflexive, and $(X,X) \in \rho_{(n)}$, i.e.

$$(x_1^{n+1}) \in X^{n+1} \subseteq \bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1}$$

If all $x_1,\dots,x_{n+1}$ are different, then $A = \{x_1^n\} \in S^{(n)}$, $B = \{x_2^{n+1}\} \in S^{(n)}$ and $(A\cup B)^{n+1} \subset \rho$. Hence, by (1_n) $(A,B) \in \rho_{(n)}$ and $(x_1^{n+1}) \in \bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1}$. Thus, $\rho \subset \bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1}$.

(ii) Let now $(x_1^{n+1}) \in \bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1}$. Then, $(x_1^{n+1}) \in (A\cup B)^{n+1}$, for some $A,B \in S^{(n)}$, such that $(A,B) \in \rho_{(n)}$. Then by (1_n) $(A\cup B)^{n+1} \subset \rho$, and $(x_1^{n+1}) \in \rho$ i.e. $\bigcup_{(A,B)\in\rho_{(n)}} (A\cup B)^{n+1} \subset \rho$. Now (i) and (ii) prove the equality (j_n).

Proposition 5. *Let* $\rho_{(n)}$ *be an equivalence relation on* $S^{(n)}$*, satisfying* (i_n) *(Lemma 2). Then the (n+1)-ary relation* ρ *on S, defined by* (j_n) *(Corollary 4), is an (n+1)-ary equivalence relation on S.*

P r o o f. (i) ρ is (1,n+1)-reflexive:

For all $a_1,\dots,a_n \in S$, $(a_1^n,a_1) \in \rho$, since there is $A \in S^{(n)}$ such that $a_1,\dots,a_n \in A$, i.e. $(a_1^n,a_1) \in A^{n+1}$, and $(A,A) \in \rho_{(n)}$.

(ii) ρ is symmetric:

If $(a_1^{n+1}) \in \rho$ then $(a_1^{n+1}) \in (A \cup B)^{n+1}$ for some $A,B \in S^{(n)}$ such that $(A,B) \in \rho_{(n)}$, and it is clear that for every $\pi \in \{1,\dots,n+1\}!$ $(a_{\pi(1)},\dots,a_{\pi(n+1)})$ also belongs to $(A \cup B)^{n+1}$ i.e. to the union in (j_n).

(iii) ρ is $n\bar{A}_1$-transitive:

Let $(a_o^n) \in \rho$ and $(a_1^{n+1}) \in \rho$, $(a_1,\dots,a_n$ are all different). Then, by (j_n) and (i_n)

$(a_o^n) \in (A \cup B)^{n+1}$ where $\{a_o^{n-1}\} \subseteq A$, $\{a_1^n\} = B$, $(A,B) \in \rho_{(n)}$.

Then also

$(a_1^{n+1}) \in (B \cup C)^{n+1}$, $\{a_2^{n+1}\} \subseteq C$, $(B,C) \in \rho_{(n)}$.

But $\rho_{(n)}$ is transitive, and thus $(A,C) \in \rho_{(n)}$. Hence for every $i \in \{1,\dots,n\}$ $(a_o^{i-1}, a_{i+1}^{n+1}) \in \rho$,[1] and of course, $(a_o^{n-1},a_{n+1}) \in \rho$.

We have thus proved that there is a bijection between the set of all (n+1)-ary equivalences on S and the set of binary equivalences on $S^{(n)}$ satisfying (i_n).

The following corollary is its application to the corresponding partitions, from which one can derive some combinatorial results (concerning, for example, the matroids [9]).

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1) In fact, we have proved that ρ satisfies ${}_i\bar{A}_1$-transitivity.

Corollary 6. *Let* ρ *be an arbitrary* $(n+1)$*-ary equivalence relation on* S, *and* $\rho_{(n)}$ *a corresponding (in the sense of* (1_n) *) binary equivalence on* $S^{(n)}$. *Let* $\phi_{\{x_1^n\}} \in S/\rho$ *where* $y \in \phi_{\{x_1^n\}} \overset{def}{\Longleftrightarrow} (y,x_1^n) \in \rho$. *Let also* $C_{\{x_1^n\}} \in S^{(n)}/_{\rho(n)}$

Then

a) $\phi_{\{x_1^n\}} = \bigcup C_{\{x_1^n\}}$;

b) $C_{\{x_1^n\}} = \phi^{(n)}_{\{x_1^n\}}$.

P r o o f. a) $y \in \phi_{\{x_1^n\}}$ iff $(y,x_1^n) \in \rho$ iff (by (1_n)) $(y,x_1{}^n) \in (A\cup B)^{n+1}$ and $(A,B) \in \rho_{(n)}$ iff (by (i_n))

$y \in A$ and $(A,\{x_1^n\}) \in \rho_{(n)}$ iff

$y \in A$ and $A \in C_{\{x_1^n\}}$ iff $y \in C_{\{x_1^n\}}$.

b) Let $A \in S^{(n)}$, Then $A \in C_{\{x_1^n\}}$ iff $(A,\{x_1^n\}) \in \rho_{(n)}$ iff $(A \cup \{x_1^n\})^{n+1} \subseteq \rho$ iff $A \subseteq \phi_{\{x_1^n\}}$ iff $A \in \phi^{(n)}_{\{x_1^n\}}$.

* * *

In the following we shall construct some closure operations on the set of all binary relations on $S^{(n)}$, and we shall prove that if we restrict one of them to the lattice of binary equivalences, the closed elements are exactly the relations satisfying (i_n).

For an arbitrary binary relation $\rho_{(n)}$ on $S^{(n)}$, $(|S| \geq n)$, define a new binary relation $\bar{\rho}_{(n)}$ on the same set $S^{(n)}$:

If $X,Y \in S^{(n)}$,

$(X,Y) \in \bar{\rho}_{(n)}$ iff there are $A,B \in S^{(n)}$ such that $X,Y \in (A\cup B)^{(n)}$ and $(A,B) \in \rho_{(n)}$.

LEMMA 7. $\rho_{(n)} \to \bar{\rho}_{(n)}$ *is a closure operation on* $P((S^{(n)})^2)$.

P r o o f.

(i) $\rho_{(n)} \subseteq \bar{\rho}_{(n)}$

Really, if $(X,Y) \in \rho_{(n)}$ then trivially $(X,Y) \in \bar{\rho}_{(n)}$, (since $X,Y \subseteq X \cup Y$).

(ii) If $\rho_{(n)} \subseteq \sigma_{(n)}$, then $\bar{\rho}_{(n)} \subseteq \bar{\rho}_{(n)}$.

Really, let $(X,Y) \in \bar{\rho}_{(n)}$. Then there are $A,B \in S^{(n)}$, such that $X,Y \in A \cup B$, and $(A,B) \in \rho_{(n)}$. Then (since $\rho_{(n)} \subseteq \sigma_{(n)}$), (A,B) $\sigma(n)$ and (X,Y) $\bar{\sigma}_{(n)}$.

(iii) $\bar{\bar{\rho}}_{(n)} = \bar{\rho}_{(n)}$;

Really, if $(X,Y) \in \bar{\bar{\rho}}_{(n)}$, then there are $A,B \in \bar{\rho}_{(n)}$, such that $X,Y \in (A \cup B)^{(n)}$ and $(A,B) \in \bar{\rho}_{(n)}$. But then there are $A_1,B_1 \in S^{(n)}$, such that $A,B \in (A_1 \cup B_1)^{(n)}$, and $(A_1,B_1) \in \rho_{(n)}$. Clearly $X,Y \in (A_1 \cup B_1)^{(n)}$ and hence $(X,Y) \in \bar{\rho}_{(n)}$.

Thus $\bar{\bar{\rho}}_{(n)} \subseteq \bar{\rho}_{(n)}$, and with (i) it gives $\bar{\bar{\rho}}_{(n)} = \bar{\rho}_{(n)}$.

LEMMA 8. *Let* $\rho_{(n)} \in P((S^{(n)})^2)$. *Then,* $\rho_{(n)}$ *satisfies* (i_n) *iff* $\bar{\rho}_{(n)} = \rho_{(n)}$.

P r o o f. If $\rho_{(n)}$ satisfies (i_n), then we have to prove the following logical equivalence:

$(X,Y) \in \rho_{(n)}$ iff there are $A,B \in S^{(n)}$ such that

(1) $\quad X,Y \in (A \cup B)^{(n)}$ and $(A,B) \in \rho_{(n)}$.

a) $\Rightarrow$ This is trivial: $A = X$ and $B = Y$;

b) $\Rightarrow$ Assume $(A,B) \in \rho_{(n)}$ and $(X,Y) \in (A \cup B)^{(n)}$. By (i_n) $(X,Y) \in \rho_{(n)}$.

Suppose now that $\bar{\rho}_{(n)} = \rho_{(n)}$. Then (1) holds. If $(A,B) \in \rho_{(n)}$, there are $A_1, B_1 \in S^{(n)}$ such that $A,B \in\in (A_1 \cup B_1)^{(n)}$ and $(A_1,B_1) \in \rho_{(n)}$. Now let X,Y be two arbitrary elements from $(A \cup B)^{(n)}$. Obviously, $X,Y \in (A_1 \cup B_1)^{(n)}$, and thus $(X,Y) \in \rho_{(n)}$, i.e. (i_n) is satisfied.

LEMMA 9. *If $\rho_{(n)}$ is an equivalence relation on $S^{(n)}$, then*

a) *$\bar{\rho}_{(n)}$ is a tolerance relation (i.e. reflexive and symmetric) on $S^{(n)}$;*

b) *If $\sigma_{(n)}$ is an equivalence relation on $S^{(n)}$, also satisfying (i_n), and if $\rho_{(n)} \subseteq \sigma_{(n)}$, then $\bar{\rho}_{(n)} \subseteq \sigma_{(n)}$.*

P r o o f. a) $\Delta \subseteq \rho_{(n)}$ (Δ-diagonal on $S^{(n)}$) since $\rho_{(n)}$ is reflexive. By Lemma 7 (i) $\rho_{(n)} \subseteq \bar{\rho}_{(n)}$ and hence $\Delta \subseteq \bar{\rho}_{(n)}$, i.e. $\bar{\rho}_{(n)}$ is reflexive.

$\bar{\rho}_{(n)}$ is symmetric: if there are $A,B \in S^{(n)}$ such that $X,Y \in (A \cup B)^{(n)}$ and $(A,B) \in \rho_{(n)}$, then clearly (X,Y) and (Y,X) both belong to $\rho_{(n)}$ (this proves that $\bar{\rho}_{(n)}$ is symmetric, even if $\rho_{(n)}$ is not).

b) By assumption $\rho_{(n)} \subseteq \sigma_{(n)}$, and hence by Lemma 7 (ii) $\bar{\rho}_{(n)} \subseteq \bar{\sigma}_{(n)}$. Since $\bar{\sigma}_{(n)} = \sigma_{(n)}$ (Lemma 8), it follows that $\bar{\rho}_{(n)} \subseteq \sigma_{(n)}$.

Since the closure $\bar{\rho}_{(n)}$ does not preserve the transitivity (see Example 1.), we shall define a new operation $\rho_{(n)} \to \overset{\Delta}{\rho}_{(n)}$ on $P((S^{(n)})^2)$, as a successive application of the closure defined above ($\bar{\rho}_{(n)}$) and the transitive closure of the binary relations ($\hat{\rho}_{(n)}$), i.e.

$$\overset{\Delta}{\rho}_{(n)} \stackrel{\text{def}}{=\!=} (\hat{\rho}_{(n)})$$

Also, when applied i times, let $\overset{\Delta}{\overset{\vdots}{\overset{\Delta}{\rho}}}\}i_{(n)} = \overset{\Delta}{\rho}{}^{i}_{(n)}$

Proposition 10. *For an arbitrary equivalence relation $\rho_{(n)}$ on $S^{(n)}$, the relation*

$$\widehat{\rho}_{(n)} = \bigcup_{i \in N} \overset{\Delta}{\rho}{}^{i}_{(n)}$$

is a minimal relation on $S^{(n)}$ which satisfies (i_n) and contains $\rho_{(n)}$ as a subrelation.

P r o o f. (x) $\widehat{\rho}_{(n)}$ is a relation on $S^{(n)}$, since for every $i \in N$ $\overset{\Delta}{\rho}{}^{i}_{(n)}$ is a subset of $(S^{(n)})^2$, and so is their union.

(xx) $\widehat{\rho}_{(n)}$ is an equivalence relation:

(r) $\Delta \subseteq \widehat{\rho}_{(n)}$, since $\Delta \subseteq \rho_{(n)} \subseteq \widehat{\rho}_{(n)}$.

(s) From Lemma 9a), and from the properties of the transitive closure it follows that $\overset{\Delta}{\rho}_{(n)}$ and hence $\widehat{\rho}_{(n)}$ preserves the symmetry of $\rho_{(n)}$.

(t) If $(X,Y) \in \widehat{\rho}_{(n)}$, $(Y,Z) \in \widehat{\rho}_{(n)}$, then for some $i,j \in N$ $(X,Y) \in \overset{\Delta}{\rho}{}^{i}_{(n)}$, $(Y,Z) \in \overset{\Delta}{\rho}{}^{j}_{(n)}$. If $k=\max\{i,j\}$ both pairs belong to the transitive relation $\overset{\Delta}{\rho}{}^{k}_{(n)}$, and thus $(X,Z) \in \overset{\Delta}{\rho}{}^{k}_{(n)}$ i.e. $(X,Z) \in \widehat{\rho}_{(n)}$.

(xxx) $\widehat{\rho}_{(n)}$ satisfies (i_n):

If $(X,Y) \in \widehat{\rho}_{(n)}$, then (X,Y) belongs to some $\overset{\Delta}{\rho}{}^{i}_{(n)}$ i.e. there are $A,B \in S^{(n)}$ such that $(A,B) \in \overset{\Delta}{\rho}{}^{i}_{(n)}$ and $X,Y \in (A \cup B)^{(n)}$. The union now has the same property.

(xxxx) $\widehat{\rho}_{(n)}$ is a minimal equivalence with the property (i_n):

Let $\rho_{(n)} \subseteq \sigma_{(n)}$, where $\sigma_{(n)}$ is an equivalence satisfying (i_n). Then by Lemma 9,b), $\bar{\rho}_{(n)} \subseteq \sigma_{(n)}$, and hence $\overset{\Delta}{\rho}_{(n)} \subseteq \sigma_{(n)}$ (the transitive closure). Itis the same with the finite application of these operators, i.e. for every $i \in N$, $\overset{\Delta i}{\rho}_{(n)} \subseteq \sigma_{(n)}$ implying $\hat{\rho}_{(n)} \subseteq \sigma_{(n)}$.

Consider now the lattice of all binary equivalences on $S^{(n)}$, $\langle E(S^{(n)}), \subseteq\rangle$.

From the construction, it follows that $\rho_{(n)} \to \hat{\rho}_{(n)}$ is a closure operation in that lattice. The partially ordered set of the closed element is (see, for example, [9]) also a lattice (in general a complete one), so called the quotient relative to that closure operation. We denote it as follows:

$$\langle \hat{E}(S^{(n)}), \subseteq \rangle \overset{def}{=\!=} \{\rho_{(n)} \mid \rho_{(n)} \subseteq E(S^{(n)}) \text{ and } \hat{\rho}_{(n)} = \rho_{(n)}\}, \subseteq \rangle .$$

(the infimum in that lattice is the same as in $\langle E(S^{(n)}), \subset\rangle$).

Let $\langle E_{n+1}(S), \subseteq \rangle$ be the lattice of all (n+1) ary equivalences on S[1)]

Corollary 11.

$$\langle E_{n+1}(S), \subseteq \rangle \cong \langle \hat{E}(S^{(n)}), \subset \rangle .$$

P r o o f. From Propositions 1,4, and 5, it follows that the mapping $f: E_{n+1}(S) \to \hat{E}(S^{(n)})$, such that $f(\rho) = \rho_{(n)}$ (where $\rho_{(n)}$ is defined in (1_n)) is 1-1 and onto. Both f and its inverse preserve the order.

Let $\rho, \sigma \in E_{n+1}(S)$, and $\rho \in \sigma$. Then

(a) $(x_1^{n+1}) \in \rho$ implies $(x_1^{n+1}) \in \sigma$.

Since $\rho = \bigcup_{(A,B)\in\rho_{(n)}} (A \cup B)^{n+1}$ and $\sigma = \bigcup_{(A,B)\in\sigma_{(n)}} (A \cup B)^{n+1}$,

(a) is equivalent with

1) The fact that this poset is a lattice is a trivial consequence of the fact that the poset of all the partitions of type n on S is a lattice.

(b) $(A,B) \in \rho_{(n)}$ implies $(A,B) \in \sigma_{(n)}$,

proving the statement.

Thus we have proved that the lattice of (n+1)-ary equivalences is up to the isomorphism contained in a lattice of binary equivalences as a quotient of a described closure.

We shall finally give an example, illustrating the application of defined closure operations on a binary equivalence, and its connection with the (n+1)-ary one.

Example 1. $S=\{a,b,c,d,e\}$, $n=2$.

$S^{(2)} = \{\{a,b\},\{b,c\},\{a,c\},\{a,d\},\{b,d\},\{c,d\},\{a,e\},$
$\{b,e\},\{c,e\},\{d,e\}$

$\rho_{(2)}$ is an equivalence relation on $S^{(2)}$ which does not satisfy the property (i_n), and the application of closure operations is illustrated by the table:

$\rho_{(2)}$	{a,b}	{b,c}	{a,c}	{a,d}	{b,d}	{c,d}	{a,e}	{b,e}	{c,e}	{d,e}
{a,b}	1	1		I						
{b,c}	1	1	I	II						
{a,c}	I		1			III				
{a,d}	I	II		1	1					
{b,d}				1	1					
{c,d}		III				1				
{a,e}							1			
{b,e}								1		
{c,e}									1	
{d,e}										1

$1:\rho_{(2)}$, $I:\bar{\rho}_{(2)}$, $II:\overset{\Delta}{\rho}_{(2)}$, $III:\overset{\bar{\Delta}}{\rho}_{(2)}=\overset{\Delta\Delta}{\rho}_{(2)}=\hat{\rho}_{(2)}$.

The corresponding ternary equivalence ρ is

$$\rho = d_2 \cup \pi(a,b,c) \cup \pi(a,b,d) \cup \pi(a,c,d) \cup \pi(b,c,d) \quad 1)$$

$$S/_{\rho} = \{\{a,b,c,d\},\{a,e\},\{b,e\},\{c,e\},\{d,e\}\}.$$

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1) $d_2 \overset{def}{=} \{(a_1^{n+1}) \mid a_1,\dots,a^{n+1} \in S$ and $a_i=a_j$ for some $i\neq j$, $i,j \in \{1,\dots,n+1\}$ (see [5]).

$\pi(x_1^{n+1}) \overset{def}{=} (x_{\pi(1)},\dots,x_{\pi(n+1)})$, $\pi \in 1,\dots,n+1\}!\{$ (see also [5]).

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REZIME

JEDNA VEZA IZMEDJU BINARNIH I (n+1)-ARNIH RELACIJA EKVIVALENCIJE NA KONAČNIM SKUPOVIMA

Uočena je veza izmedju (n+1)-arnih ekvivalencija na konačnom skupu S, i binarnih ekvivalencija na skupu $S^{(n)}$ svih n-podskupova od S.

Pokazano je da postoji izomorfizam izmedju mreže (n+1)-arnih ekvivalencija i mreže količnika po posebno konstruisanom zatvorenju na mreži svih binarnih ekvivalencija na $S^{(n)}$.