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ON THE ENUMERATION OF CYCLIC SQUARES

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ABSTRACT

A cyclic square (C-square) is a matroid with exactly four cyclic flats, which are not all in the same chain. In this paper we give formulae for the number of all, the number of self-dual and the number of binary non-isomorphic C-squares on an n-set.

PRELIMINARIES

An <u>n-set</u> is a set of cardinality n.

The cardinality of a set X is denoted by "$|X|$".

The whole part of a real number X is denoted by "$\lfloor X \rfloor$". $\lceil X \rceil = \lfloor X \rfloor + 1$.

Given $k,n \in N$, the remainder of n, when divided by k, is denoted by "$rest_k(n)$".

We assume familiarity with the notions "graph", "vertex", "edge", "cycle" (the last three notions are related to graphs), "lattice", lattice isomorphism", "chain" (type of lattice).

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An *n-cycle* is a cycle with (exactly) n edges.

An *n+cycle* is a cycle with at least n edges.

A *loop* of a graph is a 1-cycle.

A *bundle* is a set B of edges of a graph, such that each two edges of B constitute a 2-cycle.

The lattice  is denoted by "L_o".

A *matroid* M on a finite set (the ground-set of M) S is an ordered pair (S,f), where f is a function, which maps the set 2^S into itself and satisfies the following conditions for each $X,Y \subseteq S$ and for each $x,y \in S$:

(1) $X \subseteq f(X)$

(2) $X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$

(3) $f(f(X)) = f(X)$

(4) $y \in f(X \cup x) \setminus f(X) \Rightarrow x \in f(X \cup y)$

The set $f(X)$ is the *closure* of the set X.

A *flat* of M is a subset X of S, which satisfies $f(X) = X$. Two matroids are *isomorphic* if there is a bijection between their ground-sets, which preserves their flats.

Let X_o be the unique minimal flat of M (= the intersection of all flats of M) and let

$$X_o \subset X_1 \subset \ldots \subset X_h = X$$

be a chain of flats of M with the property that there does not exist a flat Y of M satisfying $X_{i-1} \subset Y \subset X_i$ for any i, $1 \le i \le h$. Given the flat X, it is well-known ([9]) that the number h does not depend on the choice of chain. We write

$$\mathrm{rank}(X) = h$$

where the function "rank" is defined only for flats.

If X is an arbitrary subset of S, then we extend the definition of rank with

$$\mathrm{rank}(X) \overset{\mathrm{def}}{=} \mathrm{rank}(f(X))$$

A subset X of S is <u>independent</u> in M if it satisfies rank(X) = |X|, otherwise it is <u>dependent</u>.

A <u>base</u> of M is a maximal independent set of M. All bases of M have the same cardinality, which is called "rank of M".

A <u>circuit</u> of M is a minimal dependent set of M.

A <u>loop</u> of M is a circuit of cardinality 1.

A flat of M is <u>cyclic</u> if it is also a union of circuits. All the cyclic flats of M constitute a lattice ([6]), ordered by inclusion, which we call the <u>CF-lattice</u> of M. Each finite lattice is the CF-lattice of a matroid ([7]).

It is well-known ([9]) that the complements with respect to the ground-set S of all bases of a matroid M are the bases of another matroid M^* on S, which is called the <u>dual</u> matroid of the matroid M. The same assertion holds when the word "bases" is replaced by "cyclic flats" ([1]). However, the cyclic flats themselves determine (uniquely up to an isomorphism) a matroid on S only provided that their ranks are given. Consequently, if we want to construct the matroid M^* by use of cyclic flats of M and their ranks, then we need the formula which connects the rank-functions of M^* and M ([9]):

$$\mathrm{rank}^*(S\setminus X) = |S| - \mathrm{rank}(S) - |X| + \mathrm{rank}(X)$$

It is obvious that $(M^*)^* \equiv M$.

A <u>coloop</u> of M is a loop of M^*.

A matroid is <u>self-dual</u> if it is isomorphic to its dual matroid.

Given a graph G, it is known ([9]) that the cycles (polygons) of G are the circuits of a matroid on the edge-set of G. This matroid is called the <u>polygon-matroid</u> of G and is denoted by "M(G)". The graph G is a <u>graphical</u>

representation of M(G). A matroid is graphic if it has a graphical representation.

If X is a subset of the edge-set of G, then rank(X) in M(G) is determined as the maximal number of edges of X which do not "completely cover" a cycle of G.

A C-chain (cyclic chain) is matroid M which satisfies:

If F_1 and F_2 are two different cyclic flats of M, then either $F_1 \subset F_2$ or $F_2 \subset F_1$ (equivalently, the CF-lattice of M is isomorphic to a chain).

A C-square (cyclic square) is a matroid which has exactly four cyclic flats, two of which are incomparable by inclusion (equivalently, the CF-lattice of which is isomorphic to L_o).

REMARK: Graphic matroids are special binary matroids (for the definition of binary matroids see, e.g., [9]). However, it is routine to show, in the same way as for C-chains in [2], that each binary C-square is also graphic. Thus a C-square is binary if and only if it is graphic.

INTRODUCTION

C-chains are considered in paper [2] and, although not explicitly, in [8] and [5], p.67. They are a very "natural" class of matroids; this can be seen from the fact that there are exactly 2^n non-isomorphic C-chains on an n-set. The number of all, the number of self-dual and the number of binary non-isomorphic C-chains on an n-set are given in [2]. It is also proved, in that paper, that all C-chains are transversal and, in addition, that all C-squares are transversal (for the definition of transversal matroids see, e.g., [4] or [9]).

This paper is a complement to paper [2]. We shall solve here three problems for C-squares, which are analogous to the first three problems solved for C-chains in [2]; that is, we shall enumerate all, self-dual and binary non-isomorphic C-squares on an n-set.

Our interest in C-squares is threefold.

Firstly, C-chains and C-squares predominate among the "small" matroids on at most 7 elements: all matroids on at most 3 elements are C-chains; all matroids on at most 5 elements are C-chains or C-squares; among 474 non-isomorphic matroids on at most 7 elements ——— only 107 are neither C-chains, nor C-squares. However, when the non-isomorphic matroids on an 8-set are considered, then the situation is completely different: there are only 256 C-chains and 219 C-squares among 1724 ([3]) such matroids.

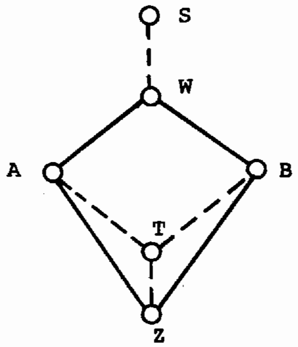
Secondly, C-squares are interesting, together with C-chains, because of their relation to transversality. The lattice L_o is the first one beside the chains of all lengths, which is known that, in the role of the "algebraic foundation" of the CF-lattice, "guarantees" the transversality of the corresponding matroids. Equivalently, chains and L_o are the first examples of so-called Tr-lattices, which are considered in the paper [4] and it seems natural to start exploring Tr-lattices exactly from these two types of lattices.

Thirdly, the enumeration of C-squares gives a good insight into the complexity of enumeration of those non-isomorphic matroids, the CF-lattices of which are isomorphic to a fixed lattice. The lattice L_o is the "simplest" lattice different from a chain. However, although the class of C-chains is by far more general and contains by far more matroids than the class of C-squares, nevertheless the enumeration of all non-isomorphic C-squares is by far more difficult than the enumeration of all non-isomorphic C-chains. Almost the same can be said for the enumeration of the self-dual matroids in these two classes.

A PARAMETRIC DESCRIPTION OF C-SQUARES

C-squares can, similarly as C-chains in [2], be completely (up to an isomorphism) determined by some numerical parameters.

Six characteristic flats, the cardinalities and ranks of which completely determine the corresponding general C-square M, are denoted by the following diagram:



The set S is the ground-set of M. The cyclic flats W and Z are 1 and 0 of the CF-lattice of M respectively, while the cyclic flats A and B are the "flanks" of that lattice. The set T is equal to $A \cap B$. The elements of the sets Z and $S \setminus W$ are the loops and the coloops of M respectively.

The denotations of the cardinalities and ranks of the six considered flats are given in the following table:

set	Z	T	A	B	W	S
cardinality	z	i	a	b	u	n
rank	0	t	p	q	w	r

The dotted lines in the diagram denote that the flats S and W, respectively, the flats T and Z, may coincide. If this is not the case, then the flat S, respectively, the flat

T, is a non-cyclic flat of the matroid M. This implies that the cardinality differences on the intervals [W,S] and [Z,T] are equal to the corresponding rank differences, which gives the equalities

$$n - u = r - w \quad \text{and} \quad i - z = t$$

We may eliminate, for example, the parameters i and u. A C-square is completely determined by the remaining nine parameters: z, t, a, p, b, q, w, n, r.

Since the "flanks" A and B are in symmetric positions, we may assume, without any loss of generality, that $p \leq q$ and that

$$p = q \quad \text{implies} \quad a \leq b.$$

These denotations will be used in the next two sections.

THE NUMBER OF ALL C-SQUARES

THEOREM 1. *The number* SQ(n) *of non-isomorphic C-squares on an n-set is given by the formula:*

$$SQ(n) = \frac{1}{161280}(n^8 + 8n^7 + 14n^6 - 28n^5 - 196n^4 - 448n^3 \pm$$

$$\begin{matrix} + 496n^2 + 1728n + 0, & \text{for } n \text{ even} \\ - 134n^2 + 468n + 315, & \text{for } n \text{ odd} \end{matrix})$$

(The coefficients of the last three summands depend on the parity of n, *while the remaining summands are common for both cases).*

P r o o f. We primarily determine the formulae for the number of non-isomorphic C-squares of fixed rank and cardinality of the ground-set, with which $S \equiv W$ (in this way we fix the parameters n,r and also w = r, after which there remain only six "partially free" parameters: z, a, b, t, p, q). We consider, in particular, the symmetric (p = q) and the non-symmetric case $(p < q)$ (thus the symmetry is based on the equality of "flank" ranks).

We denote the corresponding number of non-isomorphic C-squares with fixed n and $r = w$ by $\overline{S}_r(n)$ and $\overline{N}_r(n)$, respectively.

LEMMA 1.1 *Let* $f(n) \overset{\text{def}}{=} \sum_{j=1}^{n} \sum_{i=1}^{j} i$. *Then*

$$\overline{N}_r(n) = \sum_{\substack{p+q\geq r \\ 1\leq p<q\leq r-1}} \sum_{t=0}^{p+q-r} f(n+t-p-q-1)$$

P r o o f of the Lemma. We primarily, also, fix the parameters p,q,t and introduce the denotations

$$h_1 = a - z - (p + 1) , \quad h_2 = b - z - (q + 1)$$

The values of h_1 and h_2 are non-negative, for the cardinality difference on an interval between some two neighbouring cyclic flats in the CF-lattice must be strictly greater that the corresponding rank difference (the last observation is an easy consequence of the definitions, it should be applied to the intervals $[Z,A]$ and $[Z,B]$ respectively).

Using the relations

$$|A \cap B| + |A \cup B| = a + b ; \qquad |A \cap B| = z + t ;$$

$$|A \cup B| \leq n ; \quad a = z + p + h_1 + 1; \quad b = z + q + h_2 + 1 ,$$

we derive the inequality

$$z + h_1 + h_2 \leq n + t - p - q - 2$$

We denote shortly the right-hand side of it by "d".

Each of the parameters z,h_1,h_2 may have any integer value between 0 and d inclusively, provided that the above condition for $z+h_1+h_2$ is satisfied.

We conclude that the number F of non-isomorphic

C-squares with fixed parameters n, $r=w,p,q,t$ with which $p < q$, equals the number of nonnegative integer solutions of

$$0 \leq z + h_1 + h_2 \leq d$$

Namely, the role of the parameters a and b, for enumeration purposes, can be fully performed by the parameters h_1 and h_2. We must make a distinction between h_1 and h_2 because of the permanent assumption that $p < q$. The existence of C-squares with each particular ordered triple (z,h_1,h_2) of parameters which satisfy the above inequality follows from the fact that, when constructing a C-square with the parameters given above, each of d "free elements" may be either added to the zero, or to a "flank" or not be used at all. On the other hand, it is clear that the matroids with mutually different numerical parameters cannot be isomorphic.

An elementary calculation gives that

$$F = \sum_{j=0}^{d} \binom{j+2}{2} = \sum_{j+1=1}^{d+1} \sum_{i=1}^{j+1} i \doteq f(d+1) = f(n + t - p - q - 1)$$

We have $1 \leq p < q \leq r-1$ and because of the submodular law for the rank-function (see,e.g., [4] or [9])

$$0 \leq t \leq p+q-w = p+q-r$$

(we also have the condition $p+q \geq r$ as a consequence).

Thus the given formula for $\overline{N}_r(n)$ arises primarily by summing the expressions F for all possible values of t, while p and q are fixed and, after that, by summing over all ordered pairs (p,q) in the permissible area, which depends on r. □

LEMMA 1.2. *Let* $g(n) \overset{\text{def}}{=} \sum_{j=1}^{n} \sum_{i=1}^{j} \left\lfloor \frac{i+1}{2} \right\rfloor$

$$\overline{S}_r(n) = \sum_{p=\left\lfloor \frac{r+1}{2} \right\rfloor}^{r-1} \sum_{t=0}^{2p-r} g(n + t - 2p - 1)$$

P r o o f of the Lemma. We just have to point out the differences from the previous, non-symmetric case

If $p = q$, then $d + 1 = n + t - 2p - 1$ and $p + q - r = 2p - r$, while the area $\{(p,q) | 1 \leq p \leq q \leq r-1 \wedge \wedge p+q \geq r\}$ becomes the interval $\{p | \lfloor \frac{r+1}{2} \rfloor \leq p \leq r-1\}$.

The main difference is that the roles of the parameters h_1 and h_2 do not differ in the symmetric case. Consequently the number G of non-isomorphic C-squares with fixed parameters n, r = w, p = q, t, equals $\sum_{j=0}^{d} g_j$, where g_j is the number of different non-negative integral solutions of the equation

$$z + h_1 + h_2 = j$$

under the condition that the solutions, which can be obtained from each other just by the transposition of the values of f_1 and h_2, are considered to be equal.

$$\text{Thus} \quad g_j = \sum_{z=0}^{j} \lfloor \frac{j-z+2}{2} \rfloor = \sum_{i=1}^{j+1} \lfloor \frac{i+1}{2} \rfloor$$

$$\text{Hence} \quad G = \sum_{j+1=1}^{d+1} g_j = g(d+1) \quad ,$$

which implies the quoted formula for $\overline{S}_r(n)$. □

P r o o f of the Theorem (continued). We shall use the following denotations for the numbers of non-isomorphic C-squares of special types on an n-set:

$SQ(n)$ = the number of all C-squares
$N(n)$ = the number of non-symmetric C-squares ($p < q$)
$S(n)$ = the number of symmetric C-squares ($p = q$)
$N_r(n)$ = the number of non-symmetric rank r C-squares
$S_r(n)$ = the number of symmetric rank r C-squares
$\overline{N}_r(n)$ = the number of non-symmetric C-squares of rank r, which satisfy $S \equiv W$
$\overline{S}_r(n)$ = the number of symmetric C-squares of rank r, which satisfy $S \equiv W$.

We primarily sketch the main steps of the proof. It is clear that

$$SQ(n) = N(n) + S(n)$$

On the other hand

$$N(n) = \sum_{r=3}^{n-2} N_r(n) \qquad \text{and} \qquad S(n) = \sum_{r=2}^{n-2} S_r(n)$$

Namely, the rank of a C-square on an n-set belongs to $[2,n-2]$ (there is a positive rank increase on each elementary interval of the CF-lattice, which is strictly smaller than the corresponding cardinality increase). Both the flanks of a rank 2 C-square are of rank 1. It follows that all such C-squares are symmetric.

We further observe that

$$N_r(n) = N_{r-1}(n-1) + \overline{N}_r(n) \qquad \text{and}$$
$$S_r(n) = S_{r-1}(n-1) + \overline{S}_r(n)$$

These equalities are the consequences of the bijection between all those rank r C-squares on an n-set, which satisfy $S \neq W$ and all rank (r-1) C-squares on an (n-1)-set. This bijection is established by deletion (respectively, by addition) of one coloop.

The last two recurrence relations immediately give:

$$N_r(n) = \sum_{j=3}^{r} \overline{N}_j(n-r+j) \quad \text{and} \quad S_r(n) = \sum_{j=2}^{r} \overline{S}_j(n-r+j)$$

Finally, the formulae for $\overline{N}_r(n)$ and $\overline{S}_r(n)$ which are given by Lemmas 1.1 and 1.2, should be applied.

* * * * *

If f and g denote the auxiliary functions, which are introduced in the lemmas, then we observe primarily that for $n \in N$

$$f(n) = \frac{1}{6}(n^3 + 3n^2 + 2n) \ ;$$

$$g(n) = \frac{1}{24}(2n^3 + 9n^2 + 10n \begin{matrix} +0, & \text{for } n \text{ even} \\ +3, & \text{for } n \text{ odd} \end{matrix})$$

We denote these two polynomial formulae by $\bar{f}(n)$ and $\bar{g}(n)$, respectively.

We find and, later on, use the auxiliary sums of the forms

$$\sum_{k=1}^{n} k^j, \quad 0 \le j \le 7 \ ;$$

$$\sum_{\substack{k \text{ odd} \\ k=1}}^{n} k^j, \quad 0 \le j \le 3 \ ; \qquad \sum_{\substack{k \text{ even} \\ k=1}}^{n} k^j, \quad 0 \le j \le 3$$

(the expression for the last two types of sums depend on the parity of n).

We find the expressions $\sum_{t=0}^{p+q-r} f(n + t - p - q - 1)$ and $\sum_{t=0}^{2p-r} g(n + t - 2p - 1)$ in the developed forms.

The (developed) sums of the form $\sum_{\substack{p+q \ge r \\ 1 \le p < q \le r-1}} p^i q^j$, $0 \le i+j \le 4$, respectively, of the form $\sum_{p=\lfloor\frac{r+1}{2}\rfloor}^{r-1} p^j$, $0 \le j \le 4$, are substituted into these expressions (these later sums are among the auxiliary sums for the evaluation of the former).

Let $E = \{(p,q) \mid p,q \in N \cup \{0\} \wedge p+q \ge r \wedge \wedge 1 \le p < q \le r-1\}$

We use the following development:

$$\sum_{(p,q) \in E} p^i q^j = \sum_{p=1}^{\lfloor\frac{r-1}{2}\rfloor} p^i \sum_{k=1}^{p} (r-k)^j + \sum_{p=\lfloor\frac{r+1}{2}\rfloor}^{r-1} p^i \sum_{k=1}^{r-p} (r-k)^j .$$

In order to find 30 such sums (15 for r even and 15 for r odd), we find primarily the auxiliary sums of the form

$$\sum_{p=1}^{\lfloor \frac{r-1}{2} \rfloor} p^j \,, \qquad \sum_{p=\lfloor \frac{r+1}{2} \rfloor}^{r-1} p^j \,, \qquad 0 \le j \le 5$$

(separately for r even and r odd), and the sums of the form

$$\sum_{k=1}^{p} (r-k)^j \,, \qquad \sum_{k=1}^{r-p} (r-k)^j \,, \qquad 0 \le j \le 4$$

The evaluation of coefficients gives:

$$\overline{N}^{I}_{r}(n) = \frac{1}{2880}\Big(-84r^6 + 252r^5 - 45r^4 \substack{-\;300 \\ -\;360} r^4 \substack{+\;84 \\ +\;174} r^2 \substack{+\;48 \\ +\;108} r \substack{+\;0 \\ -\;45} +$$

$$+ 192nr^5 - 480nr^4 + 20nr^3 \substack{+\;360 \\ +\;540} nr^2 \substack{-\;32 \\ -\;212} nr \substack{+\;0 \\ -\;60} n - 150n^2r^4 +$$

$$+ 300n^2r^3 + 60n^2r^2 \substack{-\;120 \\ -\;300} n^2r \substack{+\;0 \\ +\;90} n^2 + 40n^3r^3 - 60n^3r^2 - 40n^3r \substack{+ \\ +}$$

$$\substack{+\;0 \\ +\;60} n^3 \quad \substack{, \quad r \text{ even} \\ , \quad r \text{ odd}} \Big)$$

$$\overline{S}^{I}_{r}(n) = \frac{1}{1440}\Big(-78r^5 + 165r^4 \substack{+\;30 \\ +\;60} r^3 \substack{-\;120 \\ -\;210} r^2 \substack{+\;(48,-42) \\ +\;18} r \substack{+\;0 \\ +\;45} +$$

$$+ 165nr^4 - 240nr^3 \substack{-\;60 \\ -\;150} nr^2 \substack{+\;60 \\ +\;240} nr \substack{+\;0 \\ -\;15} n - 120n^2r^3 + 90n^2r^2 \substack{+ \\ +}$$

$$\substack{+\;30 \\ +\;120} n^2r \substack{+\;0 \\ -\;90} n^2 + 30n^3r^2 \substack{+\;0 \\ -\;30} n^3 \quad \substack{, \quad r \text{ even} \\ , \quad r \text{ odd}} \Big)$$

EXPLANATIONS: If a summand has two alternatives for its coefficients, then the upper one relates to the case when r is even, while the lower one relates to the case when r is odd. If a coefficient is replaced by an ordered pair of numbers, then the first (respectively, the second) element relates to the case when n is even (respectively odd). Such denotations are also used with the formulae which follow.

The additional indices I are introduced since the above formulae for $\overline{N}_r(n)$ and $\overline{S}_r(n)$ are valid for $n \ge 2r-4$ only. The reason is that

$$\overline{f}(n) \neq f(n) = 0 \qquad \text{for } n \leq -3 \quad \text{and}$$

$$\overline{g}(n) \neq g(n) = 0 \qquad \text{for } n \leq -4 \ , \quad \text{while}$$

for relatively large values of p and q the functions f and g in the formulae for $\overline{N}_r(n)$ and $\overline{S}_r(n)$, respectively, have negative arguments smaller than -2, respectively -3.

The summands with the smallest arguments in the developments of $\overline{N}_r(n)$ and $\overline{S}_r(n)$ are $f(n-2r+2)$ and $g(n-2r+1)$, respectively. This implies that $N_r^I(n) = \overline{N}_r(n)$ and $\overline{S}_r^I(n) = \overline{S}_r(n)$ if and only if $n-2r+2 \geq -2$, respectively $n-2r+1 \geq -3$, that is, in both cases for $n \geq 2r-4$.

We define the two sequences of natural numbers:

$$w(n) \stackrel{\text{def}}{=} \frac{1}{4}\left(n^2 + 2n \begin{array}{l} +\ 0,\ n \text{ even} \\ +\ 1,\ n \text{ odd} \end{array}\right), \quad y(n) \stackrel{\text{def}}{=} \frac{1}{2}\left(n \begin{array}{l} +\ 0,\ n \text{ even} \\ +\ 1,\ n \text{ odd} \end{array}\right)$$

It is easy to show that

$$\overline{N}_r(n) = \sum_{k=r+1}^{2r-2} w(2r-1-k)f(n-k)$$

$$\overline{S}_r(n) = \sum_{k=r+1}^{2r-1} y(2r-k)g(n-k)$$

Using this, and also the relations

$$\overline{f}(-n) = -\overline{f}(n-2) \quad ; \qquad \overline{g}(-n) = -\overline{g}(n-3)$$

we find that the functions $\overline{N}_r(n)$ and $\overline{S}_r(n)$ in the area $\{(n,r) \mid n,r \in N \wedge r+2 \leq n < 2r-4\}$ should be represented by the transformed expressions $\overline{N}_r^{II}(n)$ and $\overline{S}_r^{II}(n)$, where

$$\overline{N}_r^{II}(n) = \overline{N}_r^I(n) + A_r(n) \ ; \qquad \overline{S}_r^{II}(n) = \overline{S}_r^I(n) + B_r(n) \ ,$$

while

$$A_r(n) = \sum_{j=1}^{2r-4-n} w(2r-3-n-j)\overline{f}(j)$$

$$B_r(n) = \sum_{j=1}^{2r-4-n} y(2r-3-n-j)\overline{g}(j)$$

The evaluation gives the formulae:

$$\overline{N}^{II}_r(n) = \frac{1}{2880}(44r^6 - 132r^5 + 35r^4 \substack{+180\\+120} r^3 \substack{-124\\-34} r^2 \substack{-48\\+12} r \substack{+\\+}$$

$$\substack{+(0,45)\\+(-45,0)} - 192nr^5 + 480nr^4 - 140nr^3 \substack{-360\\-180} nr^2 \substack{+176\\-4} nr \substack{+48\\-12} n +$$

$$+ 330n^2r^4 - 660n^2r^3 + 180n^2r^2 \substack{+240\\+60} n^2 r \substack{-52\\+38} n^2 - 280n^3r^3 +$$

$$+ 420n^3r^2 - 80n^3r \substack{-60\\+0} n^3 + 120n^4r^2 - 120n^4r + 5n^4 - 24n^5r +$$

$$+ 12n^5 + 2n^6 , \substack{r \text{ even}\\ r \text{ odd}})$$

$$\overline{S}^{II}_r(n) = \frac{1}{1440}(18r^5 - 75r^4 \substack{+30\\+60} r^3 \substack{+120\\+30} r^2 \substack{-48\\+(-78,12)} r \substack{+(0,-45)\\+(45,0)} -$$

$$- 75nr^4 + 240nr^3 \substack{-60\\-150} nr^2 \substack{-180\\+0} nr \substack{+(48,3)\\+(33,-12)} n + 120n^2r^3 -$$

$$- 270n^2r^2 \substack{+30\\+120} n^2 r \substack{+60\\-30} n^2 - 90n^3r^2 + 120n^3r \substack{+0\\-30} n^3 + 30n^4r -$$

$$- 15n^4 - 3n^5 \substack{,\ r \text{ even}\\ ,\ r \text{ odd}})$$

The formulae for $N_r(n)$ and $S_r(n)$ also appear in two forms: $N^I_r(n)$ and $S^I_r(n)$ for $n \geq 2r-4$, respectively $N^{II}_r(n)$ and $S^{II}_r(n)$ for $n < 2r-4$. This difference is a consequence of the corresponding difference with the formulae for $\overline{N}_r(n)$ and $\overline{S}_r(n)$. Namely:

$$N^I_r(n) = \sum_{j=3}^{r} \overline{N}^I_j(n - r + j) \quad ; \quad S^I_r(n) = \sum_{j=2}^{r} \overline{S}^I_j(n - r + j)$$

$$N^{II}_r(n) = \sum_{j=3}^{n-r+4} \overline{N}^I_j(n - r + j) + \sum_{j=n-r+5}^{r} \overline{N}^{II}_j(n - r + j)$$

$$S^{II}_r(n) = \sum_{j=2}^{n-r+4} \overline{S}^I_j(n - r + j) + \sum_{j=n-r+5}^{r} \overline{S}^{II}_j(n - r + j)$$

The evaluation gives:

$$N_r^I(n) = \frac{1}{40320}(-256r^7 + 308r^6 + 1316r^5 - 1435r^4 \begin{array}{c}-1204\\-1624\end{array} r^3 \begin{array}{c}+\\+\end{array}$$

$$\begin{array}{c}+812\\+1442\end{array} r^2 \begin{array}{c}+144\\+564\end{array} r \begin{array}{c}+0\\-315\end{array} + 616nr^6 - 504nr^5 - 2800nr^4 + 2100nr^3 \begin{array}{c}+\\+\end{array}$$

$$\begin{array}{c}+1344\\+2604\end{array} nr^2 \begin{array}{c}-336\\-1596\end{array} nr \begin{array}{c}+0\\-420\end{array} n - 504n^2r^5 + 210n^2r^4 + 2100n^2r^3 -$$

$$- 840n^2r^2 \begin{array}{c}-336\\-1596\end{array} n^2r \begin{array}{c}+0\\+630\end{array} n^2 + 140n^3r^4 - 560n^3r^2 \begin{array}{c}+0\\+420\end{array} n^3 \begin{array}{l}, \ r \text{ even}\\ , \ r \text{ odd}\end{array})$$

$$S_r^I(n) = \frac{1}{5760}(-84r^6 + 36r^5 + 375r^4 \begin{array}{c}-180\\-120\end{array} r^3 \begin{array}{c}+(-156,-246)\\+(-426,-336)\end{array} r^2 \begin{array}{c}+\\+\end{array}$$

$$\begin{array}{c}+(144,-36)\\+84\end{array} r \begin{array}{c}+0\\+(135,45)\end{array} + 192nr^5 + 30nr^4 - 640nr^3 \begin{array}{c}+180\\+0\end{array} nr^2 \begin{array}{c}+\\+\end{array}$$

$$\begin{array}{c}+88\\+448\end{array} nr \begin{array}{c}+0\\-30\end{array} n - 150n^2r^4 - 120n^2r^3 + 330n^2r^2 \begin{array}{c}-60\\+120\end{array} n^2r \begin{array}{c}+0\\-180\end{array} n^2 +$$

$$+ 40n^3r^3 + 60n^3r^2 - 40n^3r \begin{array}{c}+0\\-60\end{array} n^3 \begin{array}{l}, \ r \text{ even}\\ , \ r \text{ odd}\end{array})$$

$$N_r^{II}(n) = \frac{1}{40320}(-4n^7 + 56n^6r - 336n^5r^2 + 1120n^4r^3 - 2100n^3r^4 +$$

$$+ 2184n^2r^5 - 1176nr^6 + 256r^7 - 14n^6 + 168n^5r - 840n^4r^2 +$$

$$+ 2240n^3r^3 - 3150n^2r^4 + 2184nr^5 - 588r^6 + 56n^5 - 560n^4r +$$

$$+ 1680n^3r^2 - 2380n^2r^3 + 1680nr^4 - 476r^5 + 175n^4 - 1400n^3r +$$

$$+ 3360n^2r^2 - 3500nr^3 + 1365r^4 \begin{array}{c}-196\\+224\end{array} n^3 \begin{array}{c}+840\\-420\end{array} n^2r \begin{array}{c}-1008\\+252\end{array} nr^2 \begin{array}{c}+\\-\end{array}$$

$$\begin{array}{c}+364\\-56\end{array} r^3 \begin{array}{c}-476\\+154\end{array} n^2 \begin{array}{c}+1568\\+308\end{array} nr \begin{array}{c}-1092\\-462\end{array} r^2 \begin{array}{c}+144\\-276\end{array} n \begin{array}{c}-144\\+276\end{array} r \begin{array}{c}+(0,315), \ r \text{ even}\\+(-315,0), r \text{ odd}\end{array})$$

$$S_r^{II}(n) = \frac{1}{5760}(2n^6 - 24n^5r + 120n^4r^2 - 280n^3r^3 + 330n^2r^4 -$$

$$- 192nr^5 + 44r^6 + 6n^5 - 60n^4r + 300n^3r^2 - 600n^2r^3 + 510nr^4 -$$

$$- 156r^5 - 25n^4 + 160n^3r - 270n^2r^2 + 160nr^3 - 25r^4 \begin{array}{c}-60\\-120\end{array} n^3 \begin{array}{c}+\\+\end{array}$$

$$\begin{array}{c}+300\\+480\end{array} n^2r \begin{array}{c}-540\\-720\end{array} nr^2 \begin{array}{c}+300\\+360\end{array} r^3 \begin{array}{c}+68\\-112\end{array} n^2 \begin{array}{c}-184\\+176\end{array} nr \begin{array}{c}+(116,26)\\+(-154,-64)\end{array} r^2 \begin{array}{c}+\\+\end{array}$$

$$+ (144,54)_n - \quad 144 \quad _r + (0,-45)\ ,\quad r\ \text{even},$$
$$+ (114,24)^n + (-204,-24)^r + (135,0)\ ,\quad r\ \text{odd})$$

REMARKS: It is convenient to use the substitution $d = n - r$ when counting $N_r^{II}(n)$ and $S_r^{II}(n)$.

We observe an interesting regularity with the differences between "alternative coefficients":

The differences between the upper and the lower coefficients beside $n^i r^j$ $(i,j \in N,\ 0 \le i+j \le 3)$, coincide within each of the pairs of functions: $(\overline{N}_r^I(n),\ \overline{N}_r^{II}(n))$, $(\overline{S}_r^I(n),\ \overline{S}_r^{II}(n))$, $(N_r^I(n),\ N_r^{II}(n))$, $(S_r^I(n),\ S_r^{II}(n))$. Moreover, this coincidence extends to all the four functions beginning in "N", respectively "S" after the divisors in front of the brackets are taken into account.

* * * *

We find the formulae for $N(n)$ and $S(n)$ by summing through both the areas ($n \ge 2r-4$ and $n < 2r-4$):

$$N(n) = \sum_{r=3}^{\lfloor\frac{n+4}{2}\rfloor} N_r^I(n) + \sum_{r=\lfloor\frac{n+6}{2}\rfloor}^{n-2} N_r^{II}(n)$$

$$S(n) = \sum_{r=3}^{\lfloor\frac{n+4}{2}\rfloor} S_r^I(n) + \sum_{r=\lfloor\frac{n+6}{2}\rfloor}^{n-2} S_r^{II}(n)$$

We denote the sums in these expressions by $N^I(n)$ and $N^{II}(n)$, respectively, by $S^I(n)$ and $S^{II}(n)$.

We need the following auxiliary sums for this summation:

$$\sum_{r=3}^{\lfloor\frac{n+4}{2}\rfloor} r^j\ , \qquad \sum_{r=\lfloor\frac{n+6}{2}\rfloor}^{n-2} r^j\ , \qquad 0 \le j \le 7 \quad \text{and also}$$

$$\sum_{\substack{r\ \text{even} \\ r=2}}^{\lfloor\frac{n+4}{2}\rfloor} r_j\ , \quad \sum_{\substack{r\ \text{odd} \\ r=3}}^{\lfloor\frac{n+4}{2}\rfloor} r^j\ , \quad \sum_{\substack{r\ \text{even} \\ r=\lfloor\frac{n+6}{2}\rfloor}}^{n-2} r^j\ , \quad \sum_{\substack{r\ \text{odd} \\ r=\lfloor\frac{n+6}{2}\rfloor}}^{n-2} r^j\ , \quad 0 \le j \le 3$$

The expressions for the last four types of sums depend on $rest_4(n)$, and so the same holds for the sums $N^I(n)$, $N^{II}(n)$, $S^I(n)$, $S^{II}(n)$. However, we find it interesting that after the summing $N(n) = N^I(n) + N^{II}(n)$ and $S(n) = S^I(n) + S^{II}(n)$, the differences between the cases $rest_4(n) = 1$ and $rest_4(n) = 3$, respectively, between the cases $rest_4(n) = 0$ and $rest_4(n) = 2$, disappear completely, while the differences between the cases n even and n odd remain only with the coefficients beside $n^2, n, 1$ (this last difference with $N(n)$ only). When the formulae for $N^I(n)$, $N^{II}(n)$, $S^I(n)$, $S^{II}(n)$ are considered, however, then the differences exist with all the coefficients beside $1, n, n^2, \dots, n^7$.

We have the following formulae as a result:

$$N(n) = \frac{1}{161280}\left(n^8 + 4n^7 - 28n^6 - 98n^5 + 224n^4 + 616n^3 - 512n^2 \begin{cases} -1152n + 0, & n \text{ even} \\ -522n + 315, & n \text{ odd} \end{cases}\right)$$

$$S(n) = \frac{1}{161280}\left(4n^7 + 42n^6 + 70n^5 - 420n^4 - 1064n^3 + \begin{cases} +1008n^2 + 2880n, & n \text{ even} \\ +378n^2 + 990n, & n \text{ odd} \end{cases}\right)$$

REMARK: Almost all non-isomorphic C-squares on a fixed large ground-set are non-symmetric, since $N(n)$ is given by the formula of a higher order than the formula for $S(n)$.

The required formula for $SQ(n)$ is obtained by summing the last two formulae. □

We shall give a list of the valued of the function $SQ(n)$ for n between 1 and 24 (inclusively). These values are in order:

0, 0, 0, 1, 6, 25, 80, 219,
530, 1171, 2400, 4630, 8484, 14886, 25152, 41130,
65340, 101178, 153120, 227007, 330330, 472615, 665808, 924781.

THE NUMBER OF SELF-DUAL C-SQUARES

THEOREM 2. *There are* $\frac{1}{384}(n^4 + 4n^3 - 4n^2 - 16n)$ *non-isomorphic self-dual C-squares on an n-set, for each even* n.

P r o o f. First of all, it is obvious that there are no self-dual matroids on odd ground-sets.

Let SS(n) and NS(n), in this order, denote the numbers of non-isomorphic self-dual C-squares on an n-set, with which $a = b$, respectively $a < b$ (we could call these self-dual C-squares symmetric, respectively non-symmetric, but this time the symmetry is based on the equality of flank cardinalities). We shall show that the above partition of self-dual C-squares is in accordance with the general assumption that $p \leq q$, for the following implication holds with self-dual C-squares:

$$p \leq q \Rightarrow a \leq b$$

Let there be given a self-dual C-square, which is determined by the parameters z, t, a, b, p, q, u, n, r. The self-duality gives some additional connections among these parameters:

The complement of the zero of a C-square is the unit of the CF-lattice of the dual C-square. This gives the connection $z + u = n$, in the self-dual case, and we use it to eliminate the parameter u.

There are two possibilities for the "flanks" A and B of a self-dual C-square M on S: when establishing an isomorphism with the matroid M^*, they can be mapped (in order) either to the "flanks" $S\setminus A$ and $S\setminus B$ (Case (1)) or to the "flanks" $S\setminus B$ and $S\setminus A$ respectively (Case (2)).

Case (1): The equalities $|A| = |S\setminus A|$ and $|B| = |S\setminus B|$ give $a = b = \frac{n}{2}$

Case (2): The equality $|A| = |S\setminus B|$ implies $a + b = n$. Besides, we have

$$p = \mathrm{rank}(A) = \mathrm{rank}^*(S\setminus B) = |S| - \mathrm{rank}(S) - |B| + \mathrm{rank}(B) =$$
$$= n - \frac{n}{2} - b + q = \frac{n}{2} - b + q.$$

If $p = q$, then this case also gives $a = b = \frac{n}{2}$.

Case (2) is the only possible one for $a < b$, Case (1) is the only possible one for $a = b$ and $p < q$. For $a = b$ and $p = q$ there can be realized any of the Cases (1) and (2).

Suppose that $a > b$. Then we have Case (2) and it follows

$$a > b \Rightarrow \frac{n}{2} > b \Rightarrow \frac{n}{2} - b + q > q \Rightarrow p > q\ ,$$

which contradicts the general assumption $p \leq q$.

REMARK: The necessary equality $|A \cap B| = |(S\setminus A) \cap (S\setminus B)|$ is always satisfied, since $|A \cup B| + |A \cap B| = |A| + |B| = |S|$.

LEMMA 2.1. *For each even* $n \in N$,

$$SS(n) = \sum_{z=0}^{\frac{n}{2}-2} \sum_{\substack{1 \leq p \leq q < \frac{n}{2}-z \\ p+q \geq \frac{n}{2}-z}} \left(p + q + z - \frac{n}{2} + 1\right)$$

P r o o f of the Lemma. Since symmetric self-dual C-squares satisfy $a = b = r = \frac{n}{2}$, it follows that only four "partially free" parameters remain for a fixed n: z, p, q, t. We change primarily (one value after another) the parameter t for z, p, q fixed, then p and q for fixed z, and finally z.

The rank of the set $A \cup B$ is $\frac{n}{2} - z$. It is obvious that $t \leq p-1$ and the submodular law implies

$0 \le t \le p+q-(\frac{n}{2} - z)$. Since $q+1 \le \frac{n}{2}-z$ implies $p-1 \ge p+q-(\frac{n}{2} - z)$, it follows that the first restriction for t is superfluous and there exist $p + q - \frac{n}{2} + z + 1$ possibilities for t, when the parameters z, p and q are fixed.

It is easy to check that the only restrictions for the parameters p and q, when z is fixed, are

$$1 \le p \le q < \frac{n}{2} - z \quad \text{and} \quad p+q \ge \frac{n}{2} - z$$

Finally, $\text{rank}(W) = \frac{n}{2} - z \ge 2$ implies $0 \le z \le \frac{n}{2} - 2$ (condition $z \le n-4$, which can be derived by comparing cardinalities, is superfluous, for $n \ge 4$ implies $\frac{n}{2} - 2 \le n-4$). This completes the formula for $SS(n)$. □

LEMMA 2.2. *For each even* $n \in N$,

$$NS(n) = \sum_{z=0}^{\frac{n}{2}-3} \sum_{a=z+2}^{\frac{n}{2}-1} \sum_{p=\lceil\frac{a-z}{2}\rceil}^{a-z-1} (2p - a + z + 1)$$

P r o o f of the Lemma. Since the equalities $b = n - a$, $q = p - \frac{n}{2} + b = p + \frac{n}{2} - a$, $w = n - z$ and $r = \frac{n}{2}$ are valid in the non-symmetric case, we can again leave only four "partially free" parameters, for example, z, a, p, t. We change primarily the parameter t for z, a, p fixed, then p for fixed z and a, after that a for fixed z and finally z.

The restrictions for t are $t \le p-1$ and $0 \le t \le p+q - (\frac{n}{2} - z) = p + (p - a + \frac{n}{2}) - \frac{n}{2} + z = 2p - a + z$

Since $a-z \ge p+1$ implies $2p-a+z \le p-1$, it follows that the first restriction for t is superfluous. Thus there exist $2p - a + z + 1$ possibilities for t, when the parameters z, p, a are fixed.

The used restriction for t has the inequality $0 \le 2p-a+z$ as a consequence, which implies $p \ge \frac{a-z}{2}$. The parameter p also satisfies the restriction $p \le a-z-1$ (because of $\text{rank}(A) < |A\setminus Z|$).

Since $a < b$ implies $a < \frac{n}{2}$, we conclude that the only restrictions for a, when z is fixed, are

$$z + 2 \leq a \leq \frac{n}{2} - 1$$

The last prolonged inequality implies the restriction $0 \leq z \leq \frac{n}{2} - 3$ for the parameter z. The Lemma is proved. □

In what follows we shall transform the expressions given in Lemmas 2.1 and 2.2 into formulae of a polynomial type:

LEMMA 2.3.

$$SS(n) = \frac{1}{768}\left(n^4 + 8n^3 + 8n^2 - 32n \begin{matrix} +\ 0, & \text{for } (n/2) \text{ even} \\ -\ 48, & \text{for } (n/2) \text{ odd} \end{matrix}\right)$$

for each natural number n.

P r o o f of the Lemma. We introduce the substitution $d = \frac{n}{2} - z$. We denote

$$D = \{(p,q) \mid p,q \in N \wedge 1 \leq p \leq q < d \wedge p+q \geq d\}$$

Then $SS(n) = \sum_{d=2}^{\frac{n}{2}} \sum_{D} (p + q - d + 1)$

We find $\sum_{D} 1 = \frac{1}{4}\left(d^2 \begin{matrix} +\ 0, & d \text{ even} \\ -\ 1, & d \text{ odd} \end{matrix}\right)$

$$\sum_{D} p = \frac{1}{8}\left(d^3 \begin{matrix} +\ 0, & d \text{ even} \\ -\ d, & d \text{ odd} \end{matrix}\right), \quad \sum_{D} q = \frac{1}{24}\left(5d^3 - 3d^2 \begin{matrix} -\ 2d + 0, & d \text{ even} \\ -\ 5d + 3, & d \text{ odd} \end{matrix}\right)$$

This implies

$$\sum_{D} (p + q - d + 1) = \frac{1}{24}\left(2d^3 + 3d^2 - 2d \begin{matrix} +\ 0, & d \text{ even} \\ -\ 3, & d \text{ odd} \end{matrix}\right)$$

The summation with respect to d gives the result. □

LEMMA 2.4.

$$NS(n) = \frac{1}{768}\left(n^4 - 16n^2 \begin{matrix} +\ 0, & \text{for } (n/2) \text{ even} \\ +\ 48, & \text{for } (n/2) \text{ odd} \end{matrix}\right)$$

for each even natural number n.

P r o o f of the Lemma. We introduce the substitution $h = a - z - 1$ in the expression for NS(n), which is given in Lemma 2. Thus the expression is transformed into

$$\sum_{z=0}^{\frac{n}{2}-3} \sum_{h=1}^{\frac{n}{2}-z-2} \sum_{p=\lceil\frac{h+1}{2}\rceil}^{h} (2p - h)$$

We denote $\sum\limits_{p=\lceil\frac{h+1}{2}\rceil}^{h}$ by $\sum_1$, $\sum\limits_{h=1}^{\frac{n}{2}-z-2}$ by $\sum_2$ and $\sum\limits_{z=0}^{\frac{n}{2}-3}$ by $\sum_3$.

Using the auxiliary sums:

$$\sum_1 1 = \frac{1}{2}\left(h \begin{matrix} + 0 , & h \text{ even} \\ + 1 , & h \text{ odd} \end{matrix}\right); \quad \sum_1 p = \frac{1}{8}\left(3h^2 \begin{matrix} + 2h & + 0 , & h \text{ even} \\ + 4h & + 1 , & h \text{ odd} \end{matrix}\right)$$

$$\sum_2 1 = \frac{1}{2}(n - 4) - z ;$$

$$\sum_2 h = \frac{1}{8}(n^2 - 6n + 8) + \frac{1}{2}(-n + 3)z + \frac{1}{2}z^2 ;$$

$$\sum_2 h^2 = \frac{1}{24}(n^3 - 9n^2 + 26n - 24) + \frac{1}{12}(-3n^2 + 18n - 26)z + $$

$$+ \frac{1}{2}(n - 3)z^2 - \frac{1}{3}z^3 ;$$

$$\sum_3 1 = \frac{1}{2}(n - 4) ; \quad \sum_3 z = \frac{1}{8}(n^2 - 10n + 24) ;$$

$$\sum_3 z^2 = \frac{1}{24}(n^3 - 15n^2 + 74n - 120)$$

$$\sum_3 z^3 = \frac{1}{64}(n^4 - 20n^3 + 148n^2 - 480n + 576) ,$$

we obtain the above formula for NS(n) simply. □

The proof of Theorem 2 is completed by summing the expressions given in Lemmas 2.3 and 2.4.

The differences between the cases $rest_4(n) = 0$ and $rest_4(n) = 2$ disappears after this summing and so we obtain the unique polynomial formula for each even n. □

REMARK: We observe that NS(n) = SS(n-2) for each even $n \geq 2$. It would be interesting to establish the corresponding one-to-one correspondence.

THE NUMBER OF GRAPHIC (= BINARY) C-SQUARES

THEOREM 3. *There are* $\frac{1}{2}(n^3 - 8n^2 + 21n - 18)$ *non--isomorphic graphic (= binary) C-squares on an n-set.* $(n \geq 2)$

LEMMA 3.1. *The polygon-matroid of graph* G *is a C-square if and only if all the 2+cycles of the graph* G *appear in one of the following combinations:*

a) *two edge-disjoint bundles with* x *and* y *edges, respectively*

b) *one 3-cycle, two edges of which are replaced by bundles with* x *and* y *edges, respectively*

c) *one bundle with* x *edges and one 3+cycle with* y *edges, which are mutually edge - disjoint*

d) *two edge-disjoint 3+cycles with* x *and* y *edges, respectively*

e) *two 3+cycles with* x *and* y *edges respectively, which have exactly one edge in common.*

P r o o f of the Lemma. Cases a) and b) correspond to the situation when both "flanks" are of rank 1. The closure of their union is a rank 2 cyclic flat. This closure is in the graphic case either equal to the union of the "flanks" (case (a)) or has just one element more (case (b)).(*)

Case c) relates to the situation when one "flank" is of rank 1, while the other one is of a higher rank.

If both "flanks" are of ranks higher than 1, then each of the corresponding sets contains, beside the loops, the edges of a 3+cycle. Let these two cycles be denoted by C_1 and C_2 and the set of all loops by Z. The cycles C_1 and C_2 either have no common edges (case (d)) or have exactly one common edge (case (e)). Namely, if C_1 and C_2

(*) *the only possibility to add an edge to a set of edges without raising the rank of the set is to close a cycle.*

have more than one common edge, then the set $((C_1 \cup C_2)\setminus(C_1 \cap C_2)) \cup Z$ is a cyclic flat, which is not comparable with any of the cyclic flats $C_1 \cup Z$ and $C_2 \cup Z$. In this case, however, the corresponding polygon-matroid is not a C-square. □

P r o o f of the Theorem. Let a graph G, with n edges, have z loops and t edges, which do not belong to any cycle. These two kinds of edges, respectively, correspond to the loops and coloops of the polygon-matroid M(G). If x and y have the same meaning as in the assertion of Lemma 3.1., then it is easy to show that the number of non-isomorphic graphic C-squares on an n-set, which correspond to the cases a) - e), is equal to the number of (different) integral non-negative solutions of the following equations, which satisfy the corresponding restrictions:

	equation	restrictions
a)	$x + y + z + t = n$	$x \geq 2$, $y \geq x$
b)	$x + y + z + t = n - 1$	$x \geq 2$, $y \geq x$
c)	$x + y + z + t = n$	$x \geq 2$, $y \geq 3$
d)	$x + y + z + t = n$	$x \geq 3$, $y \geq x$
e)	$x + y + z + t = n + 1$	$x \geq 3$, $y \geq x$

Case c) is unique in that it is the only case in which the variables x and y play different roles. It follows that they are mutually independent, which immediately gives that the number of solutions in that case is equal to $\binom{n-2}{3}$.

We denote the number of different solutions in case a) by $F(n)$. It is easy to check that the numbers of different solutions in cases b), d) and e) are given by $F(n-1)$, $F(n-2)$, $F(n-1)$, respectively. In order to determine the function $F(n)$, we divide $\binom{n-1}{3}$ integral non-negative solutions to the equation $x + y + z + t = n$, with which $x \geq 2$ and $y \geq 2$, into three pairwise disjoint classes, depending on which of the relations $x > y$, $x = y$ and $x < y$ is satisfied. Let the number of different solutions in these classes be denoted by $K(n)$, $L(n)$, $K(n)$, respectively. It is easy to find that

$$L(n) = \frac{1}{4}(n-2)^2 \quad \text{for } n \text{ even and}$$

$$L(n) = \frac{1}{4}(n-1)(n-3) \quad \text{for } n \text{ odd. Hence we have}$$

$$F(n) = K(n) + L(n) = \frac{1}{24}(2n^3 - 9n^2 + 10n \begin{matrix} +0, & \text{for } n \text{ even} \\ -3, & \text{for } n \text{ odd} \end{matrix})$$

According to Lemma 3.1., the required number of non--isomorphic graphic C-squares on an n-set can be found as

$$\binom{n-2}{3} + F(n) + 2F(n-1) + F(n-2)$$

This sum does not depend on the parity of n and equals the polynomial in the assertion of Theorem 3. □

REMARK: It is easy to prove that $K(n) = F(n-1)$, $n \geq 1$.

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REZIME

O PREBRAJANJU CIKLIČKIH KVADRATA

Ciklički kvadrat (C-kvadrat) je matroid sa tačno četiri ciklička potprostora, koji nisu svi u istom lancu. U ovom radu dajemo formule za broj svih, broj samodualnih i broj binarnih neizomorfnih C-kvadrata na n-skupu.