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A TRICHOTOMY OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The main result of this paper is contained in the Lemma, in which a sufficient condition is given such that all positive decreasing solutions of equation $y'' = f(x)y$ are slowly or rapidly or regularly varying functions. By using the Lemma some inequalities for the solution of the above equation are obtained.

1. INTRODUCTION

The asymptotics of solutions of the equation

$$(1.1) \qquad y'' = f(x)y$$

with $f(x)$ continuous and positive for $x > 0$ has been thoroughly studied by a large number of authors (Cf. e.g. [1], [2], [3]). Among the wealth of results, we quote the following three as typical - at least for our purposes:

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PROPOSITION 1. ([1, Ch. IV, Th. 14]).
If

$$\int_a^\infty |f^{-3/2}(t)f''(t)|dt < \infty$$

then the equation (1.1) has a fundamental system of solutions satisfying, for $x \to \infty$,

$$(1.2)\qquad \begin{aligned} y(x) &\sim f^{-1/4}(x)\exp(\pm\int_a^x f^{1/2}(t)dt) \\ y'(x) &\sim f^{1/4}(x)\cdot\exp(\pm\int_a^x f^{1/2}(t)dt) . \end{aligned}$$

PROPOSITION 2. ([2, Ex. 9.9.b]).
If

$$\int_a^\infty t^{2p-1}f^p(t)dt < \infty$$

for some $p \in [1,2]$ *then the equation (1.1) has a fundamental system satisfying, for* $x \to \infty$,

$$(1.3)\qquad \begin{aligned} y(x) &\sim \exp(-\int_a^x t\,f(t)dt), \quad \frac{y'(x)}{y(x)} = O(1/x) \\ y(x) &\sim t\exp(\int_a^x t\,f(t)dt), \quad \frac{y'(x)}{y(x)} \sim 1/x. \end{aligned}$$

PROPOSITION 3. ([4, Satz 23]).
Let $\alpha > 0$; *if*

$$\int_a^\infty t\left|f(t) - \frac{\alpha}{t^2}\right| dt < \infty$$

then equation (1.1) has a fundamental system of solutions satisfying for $x \to \infty$

$$y(x) \sim x^{\beta_i}, \quad y'(x) \sim \beta_i x^{\beta_i - 1}, \quad i = 1,2,$$

where β_i *are roots of the equation* $r(r-1) = \alpha$.

Obviously, the conditions of Proposition 1 - 3 differ strongly among each other but lead to the same type of conclusions. Is there any intrinsic reason for that or, in another words, what would be the relations - if any - between the classes of functions satisfying the conditions? It is also a natural question whether one could equalize the requirement imposed on function $f(x)$ in the above propositions by relaxing these, yet still obtain some information about the asymptotics of solutions.

At it is pointed out by Omey [5], the behavior of $x^2f(x)$ is crucial in that respect. (For a result in a somewhat similar direction, cf. Read [6]).

We shall also assume that $\lim\limits_{x\to\infty} x^2f(x) = c$ exists as a finite or infinite one, and show that the set of all positive decreasing solutions of (1.1) can be split into three disjoint subsets, according as $c = 0$, $c = \infty$, or $c \in (0,\infty)$.

The properties pertinent to asymptotics of functions belonging to one of these classes differ essentially from the ones belonging to the other two. This enlightens the first problem. As for the second, we derive, using the previous fact, some asymptotic properties of the solutions in question.

To formulate our results we need the notion of slowly and regularly varying functions as introduced by Karamata [7] and of rapidly varying functions as introduced by Bekessy [8]. These classes of functions are of increasing use in analysis and in stochastic processes in general.

DEFINITION 1. *A positive continuous function* L *defined on* (a,∞) *is said to be slowly varying (s.v.f) at infinity if for all* $t > 0$

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 .$$

DEFINITION 2. *A function* g *of the form*

$$g(x) = x^{\alpha}L(x), \quad \alpha \text{ real}$$

is said to be regularly varying (o-r.v.f) at infinity with the regularity index α.

DEFINITION 3. *A positive continuous function* g *defined on* (a,∞) *is said to be rapidly varying (r.v.f) at infinity if for all* $t > 1$

$$\lim_{x\to\infty} \frac{g(tx)}{g(x)} = 0, \quad or = \infty.$$

Positive functions tending to positive constants or iterated logarithmic functions are examples of s.v.f and interated exponential functions are the ones of r.v.f The fundamental properties of s.v.f and of o-r.v.f can be found in [9] and those of the r.v.f in [8]. Among these we need the following

PROPOSITION 4. *Let* g(x) *be positive and differentiable, for* $x > 0$ *and put*

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \ell;$$

then g(x) *is slowly, regularly, rapidly, varying according as* $\ell = 0$, $\ell = c$, $(0 < c < \infty)$, $\ell = \pm\infty$.

DEFINITION 4. *A function* g(x) *is said to be almost increasing if there exists a constant* $A > 1$ *such that* $x_2 < x_1$ *imlies* $g(x_2) \leq Ag(x_1)$; *almost decreasing functions are defined likewise.*

The fact that g(x) is almost increasing to infinity or almost decreasing to zero for $x \to \infty$ will be denoted by $g(x) \nearrow\!\!\nearrow \infty$, $g(x) \searrow\!\!\searrow 0$, respectively.

Throughout the paper all minorizing constants are denoted by the same letter m, and the majorizing ones by M, unless their exact values are needed.

2. RESULTS

We consider positive decreasing solutions y(x) of (1.1), which always exist [10 Th. 1]. Since f(x) > 0, y(x) are also convex and $y(x) \to \ell$ $(\ell \geq 0)$ and $y'(x) \to 0$.

The following result is fundamental:

LEMMA. *Let*

$$x^2 f(x) \to c, \quad x \to \infty \tag{2.1}$$

then all positive decreasing solutions of (1.1) are slowly or rapidly or regularly varying functions, with index

$$\alpha = \frac{1-\sqrt{1+4c}}{2}$$

in the later case, according as

a) $c = 0$, *b)* $c = \infty$, *c)* $c \in (0,\infty)$.

PROOF. a) By integrating both sides of equation (1.1) over (x,∞), using (2.1), and since y(x) is decreasing, one obtains for each $\varepsilon > 0$ and $x \geq x_0(\varepsilon)$

$$0 < \frac{-xy'(x)}{y(x)} \leq \varepsilon$$

and Proposition 4 applies with $\ell = 0$.

b) From (1.1) we derive, using (2.1), b)

$$y'^2(x) \geq 2 \int_x^{kx} f(t)y(t)(-dy) \geq \frac{2\Delta_1}{kx^2}\, y^2(x)\left(1 - \left(\frac{y(kx)}{y(x)}\right)^2\right) \tag{2.2}$$

with an arbitrary large Δ_1 and for $x > x_0(\Delta_2)$. On the other hand we have

$$-y'(x) \geq y(kx) \int_x^{kx} f(t)dt \geq \Delta_2 y(kx)\left(\frac{1}{x} - \frac{1}{kx}\right) \tag{2.3}$$

Now, for each sequence $\{x_n\}$ such that $y(kx_n) = O(y(x_n))$, inequality (2.2) implies for $x > x_0$

$$\frac{-xy'(x)}{y(x)} > \Delta \tag{2.4}$$

where Δ is arbitrarily large. Also for any other sequence $\{\tilde{x}_n\}$ for which, on the contrary, there holds $y(k\tilde{x}_n) \geq my(\tilde{x}_n)$, $m > 0$, the inequality (2.3) leads to the same conclusion as before.

Thus for all x, there follows $-xy'(x)/y(x) \to \infty$, $x \to \infty$ and Proposition 4 applies with $\ell = \infty$.

c) We first show that $y(x)$ is regularly varying and then determine the index.

First of all $-xy'(x)/y(x)$ is bounded. By using (2.1), c) and arguing as in part b), one obtains for $x > x_0$

$$\frac{-xy'(x)}{y(x)} \leq c + \varepsilon.$$

On the other hand

$$y'^2(x) \geq (c - \varepsilon)\int_x^\infty \frac{y(t)}{t^2}(-y'(t))dt, \quad x \geq x_0$$

or by partial integration, and since $y(x)$ decreases

$$y'^2(x) \geq (c - \varepsilon)\left\{\frac{y^2(x)}{2x^2} - \frac{y(x)}{x}\int_x^\infty \frac{y(x)}{t^2}\, dt\right\}.$$

But

$$-\int_x^\infty \frac{y(t)}{t^2}dt \geq (c - \varepsilon)y'(x)$$

so that the previous inequality reduces to

$$y'^2(x) - \frac{y(x)}{2x}y' - \frac{c-\varepsilon}{2}\frac{y^2(x)}{x^2} > 0.$$

Hence, $y'(x)$ being negative,

$$\frac{-xy'(x)}{y(x)} > \frac{1}{2}((2c + 1 - \varepsilon)^{1/2} - 1) > 0, \quad x \geq x_0.$$

Thus the quotient under consideration is bounded, and if it is also monotone, it tends to a positive limit. If it oscillates, we use the following device of Omey (*loc. cit.*): From (1) there follows

$$x\left(\frac{-xy'}{y}\right)' = \left(\frac{xy'}{y}\right)^2 - \left(\frac{xy'}{y}\right) - x^2f(x)$$

so that at the points of extrema x_n of $-xy'(x)/(y(x))$ one has

$$\left(\frac{xy'}{y}\right)^2 - \left(\frac{xy'}{y}\right) - x^2f(x) = 0.$$

Thus, because of (2.1), c) for $x = x_n$ there holds

$$0 < \frac{-xy'}{y} \to \frac{\sqrt{1+4c} - 1}{2}, \quad n \to \infty;$$

again $-xy'(x)/y(x)$ tends to a positive limit for $x \to \infty$. Whence, by Proposition 4, $y(x)$ is regularly varying of index $-\alpha$ which is negative since $y(x) \to 0$; accordingly, by Definition 2,

$$(2.5) \qquad y(x) = x^{-\alpha}L(x).$$

To determine the index we proceed as follows:

$$-y'(x) = x^{-1-\alpha}\int_1^\infty t^{-2-\alpha} L_1(tx)dt, \quad x \to \infty,$$

where $L_1(z) = z^2f(z)L(z)$ or by applying Th. 2.6 of [9]

$$-y'(x) \sim \frac{-c}{\alpha+1} x^{-1-\alpha}L(x), \quad x \to \infty.$$

By repeating the procedure one gets

$$y(x) \sim -\frac{c}{\alpha(\alpha+1)} x^{-\alpha}L(x), \quad x \to \infty$$

and so $\alpha(\alpha+1) = c$, due to (2.5). The positive root of this equation gives the wanted index.

REMARK. The corresponding linearly independent solutions $\tilde{y}(x)$ tend to infinity for $x \to \infty$ in all cases a), b), c).

In the first one, these are the regularly varying of index one, in the second one of index $1 + \sqrt{1+4c}$. This is obtained by using

$$\tilde{y}(x) = y(x) \int_a^x \frac{dt}{y^2(t)}$$

and by applying the L'Hospital rule to the quotient

$$\frac{\int_a^x \frac{dt}{y^2(t)}}{\frac{x}{y^2(x)}} .$$

In the third case, $\tilde{y}(x)$ are again rapidly varying which is obtained as in the case of $y(x)$, with only the first part of the proof needed.

2.1. Asymptotic estimates. It is now possible to obtain, almost as corollaries of the Lemma, some inequalities for solutions of (1.1) which are less precise than (1.2) and (1.3), but require milder hypotheses.

THEOREM 1. *For any positive decreasing solution* $y(x)$ *of (1.1), the condition*

$$x^2 f(x) \searrow 0, \quad x \to \infty \tag{2.6}$$

implies that there exist $a > x_0$, $\delta > 0$ *such that*

$$\exp\left(-\int_a^x \int_t^\infty f(u)du\, dt\right) < y(x) < \exp\left(-(1-\delta)\int_a^x \int_t^\infty f(u)du\, dt\right) \tag{2.7}$$

and condition

$$x^2 f(x) \to \infty, \quad x \to \infty \tag{2.8}$$

implies that there exist $a \geq x_0, \delta > 0$ *such that*

$$y(x) \leq \exp\left(-M \int_a^x \sqrt{f(t)}\, dt\right). \tag{2.9}$$

PROOF. The left hand side inequality in (2.7) follows trivially by integrating

$$\frac{-y'(x)}{y(x)} \leq \int_x^\infty f(t)dt$$

over (a,x).

To prove the right hand side one, notice that (2.6) implies $k\, f(kx)/ f(x) < A/k$, whence for sufficiently large k

$$\int_x^{kx} f(t)dt = \int_x^\infty f(t) \left\{1 - k \frac{f(kt)}{f(t)}\right\}dt > \left(1 - \frac{A}{k}\right)\int_x^\infty f(t)dt. \tag{2.10}$$

On the other hand,

$$\frac{-y'(x)}{y(x)} \geq \frac{y(kx)}{y(x)} \int_x^{kx} f(t)\, dt. \tag{2.11}$$

But $y(x)$ is, due to Lemma 1, a s.v.f so that $y(kx)/ y(x) \to 1$, $x \to \infty$ so that (2.10) and (2.11) together give for $x \geq x_0$

$$\frac{-y'(x)}{y(x)} \geq \left(1 - \frac{A}{k}\right)(1 + \varepsilon) \int_x^\infty f(t)dt$$

and the result follows by integrating over (a,x) as before.

Inequality (2.9) is an easy consequence of the Lemma. For,

$$y'^2(x) \geq \int_x^{kx} f(t)y(t)(-y'^2(t))dt > mf(x)y^2(x) \left\{1 - \frac{y^2(kx)}{y^2(x)}\right\}.$$

Thus

$$\frac{-y'(x)}{y(x)} \geq m\sqrt{f(x)}, \quad x > x_0,$$

since $y(kx)/y(x) \to 0$, $x \to \infty$, $y(x)$ being a r.v.f due to the Lemma. The result now follows by integrating the last inequality over (a,x).

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REZIME

O TRIHOTOMIJI REŠENJA LINEARNE DIFERENCIJALNE JEDNAČINE DRUGOG REDA

Osnovni rezultat ovog rada je sadržan u lemi, u kojoj je dat dovoljan uslov takav da sva pozitivna opadajuća rešenja jednačine $y'' = f(x)y$ su sporo ili brzo ili regularno promenljive funkcije. Korišćenjem leme dokazane su neke nejednakosti za rešenje gornje jednačine .

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