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ON GLOBAL RANDOM SOLUTIONS FOR RANDOM INTEGRAL AND DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT

In this papaer, we obtain existence, uniqueness and approximation results of global random solutions for random non-linear integral and differential equations. The method used in this paper differs from that used in [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

1. INTRODUCTION

It is well-known that the theory of random integral and differential equations has widespread applications to many practical problems.For applications in hereditary mechanics, telephone traffic theory, turbulence theory, population dynamics, stochastic control, biology, chemical kinetics, etc., the reader may consult [1 - 3]. Recently, Engl [3,4], Ding [5,6], Lee & Padgett [7,8], Itoh [9], and De Blasi & Myjak [10] have given some existence theorems of random solutions for random nonlinear integral and differential equations.

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2. PRELIMINARIES

Let $(X,\|\cdot\|)$ be a separable Banach space and (Ω,A,μ) be a complete σ-finite measure space. A mapping $x:\Omega \to X$ is said to be weak measurable (resp. measurable), if for any open (resp. closed) subset B of X, $x^{-1}(B) = \{\omega : x(\omega) \in B\} \in A$. Obviously, in our setting, the weak measurability is equivalent to the measurability (cf., e.g., [11]).

Let $I = [t_0, t_0+a]$ be a nondegenerate interval of the real line R. We say that $u:\Omega \times I \to X$ is a Carathéodory function if for each $t \in I$, $u(\cdot,t)$ is measurable and, for each $\omega \in \Omega$, $u(\omega,\cdot)$ is continuous. Analogously, $K:\Omega \times I \times I \times X \to X$ is a Carathéodory function, if for each $(t,s,x) \in I \times I \times X$, $K(\cdot,t,s,x)$ is measurable and, for each $\omega \in \Omega$, $K(\omega,\cdot,\cdot,\cdot)$ is continuous.

Let $C(I,X) = \{x: I \to X \mid x \text{ is continuous}, \|x\|_I = \max_{t\in I}\|x(t)\|$. Then $(C(I,X), \|x\|_I)$ is a separable Banach space. Let $C(\Omega \times I,X)$ denote the set of all Carathéodory functions $u:\Omega \times I \to X$ and $C(\Omega \times I \times I \times X,X)$ denote the set of all Carathéodory functions $K:\Omega \times I \times I \times K \to X$.

LEMMA 2.1. [9]. *A function* $u \in C(\Omega \times I,X)$ *if and only if the function* $\omega \to u(\omega,t)$, *as a function from* Ω *into* $C(I,X)$, *is measurable, i.e.* $u(\cdot,t)$ *is a* $C(I,X)$*-valued measurable function.*

LEMMA 2.2. [1]. *Let* $T:\Omega \times X \to X$ *be a Charathéodory function and let* $x:\Omega \to X$ *be measurable. Then* $T(\cdot,x(\ ))\colon\Omega \to X$ *is a X-valued measurable function.*

As a special case of Lemma 4.3. of [5], we have

LEMMA 2.3. *Let* $K \in C(\Omega \times I \times I \times X,X)$. *Then for any* $u \in C(\Omega \times I,X)$, *the function*

$$\int_{t_0}^{t} K(\cdot,t,s,u(\cdot\ s))ds$$

is $C(I,X)$*-valued measurable.*

LEMMA 2.4. *Let* $x_0 \in C(\Omega \times I, X)$ *and* $K \in C(\Omega \times I \times I \times X, X)$. *Then the random nonlinear integral operator defined by*

$$(2.1) \qquad T(\omega, x(\omega,t)) = x_0(\omega,t) + \int_{t_0}^{t} K(\omega,t,s,x(\omega,t))ds$$

satisfies the following conditions:

(i) for each $x \in C(\Omega \times I, X)$, $T(\cdot, x(\cdot,t))$ *is a* $C(I,X)$-*-valued measurable,*

(ii) for each $\omega \in \Omega$, $T(\omega,\cdot): C(\Omega \times I,X) \to C(\Omega \times I,X)$ *is continuous under the norm* $\|\cdot\|_I$, *whenever for all* $(t,s,x) \in I \times I \times X$, $\|K(\omega,t,s,x)\| \leq M(\omega)$ *where* $M:\Omega \to (0,\infty)$ *is a given function.*

PROOF. Since $x \in C(\Omega \times I,X)$ and $K \in C(\Omega \times I \times I \times X,X)$, from Lemma 2.1 and 2.3 it follows that conclusion (i) holds. By the definition of T, we obviously have $T(\omega,\cdot): C(\Omega \times I,X) \to C(\Omega \times I,X)$. Now suppose that $\{x_n\}_{n\geq 0} \subset C(\Omega \times I,X)$ and for each $\omega \in \Omega$, $\{x_n(\omega,s)\}_{n\geq 0}$ converges to $x^*(\omega,s)$ under the norm $\|\cdot\|_I$. By the continuity of K, we have $K(\omega,t,s,x_n(\omega,s)) \to K(\omega,t,s,x^*(\omega,s))$ for each $\omega \in \Omega$. Since $\|K(\omega,t,s,x_n(\omega,s))\| \leq M(\omega)$ for each $\omega \in \Omega$, an easy application of the bounded convergence theorem shows that for each $\omega \in \Omega$,

$$\int_{t_0}^{t} K(\omega,t,s,x_n(\omega,s))ds \to \int_{t_0}^{t} K(\omega,t,s,x^*(\omega,s))ds$$

under the norm $\|\cdot\|_I$. Hence conclusion (ii) of Lemma 2.4 holds.

Let (X,d) be a complete metric space and $T: X \to X$. For each $x \in X$, $O_T(x,0,\infty) = \{T^n x: n \geq 0\}$ denotes the orbit of T at x and, for any $B \subset X$, $D_d(B) = \sup\{d(x,y): x,y \in B\}$ denotes the diameter of B.

LEMMA 2.5. *Let* T *be a continuous self-mapping. Suppose that for each* $x \in X$, $D_d(O_T(x,0,\infty)) < \infty$. *If there exist positive integers* p,q *and a real number* $\beta \in (0,1)$ *such that for all* $x,y \in X$

$$(2.2) \qquad d(T^p x, T^q y) \leq \beta D_d(O_T(x,0,\infty) \cup O_T(y,0,\infty)).$$

Then for each $x \in X$, $\{T^n x\}_{n\geq 0}$ *converges to a unique fixed point* x^* *of* T.

PROOF. Letting $\Phi(t) = \beta t$, $\beta \in (0,1)$ in Corollary 6 of [12], we obtain, immediately, the conclusion of Lemma 2.5.

3. EXISTENCE UNIQUENESS CRITERIA OF GLOBAL RANDOM SOLUTIONS

In this section, we shall give the existence uniqueness and approximation theorems of global random solutions for random nonlinear Volterra integral equations and the Cauchy problem of random nonlinear differential equations.

The random nonlinear Volterra integral equation under consideration has the form

$$x(\omega,t) = x_0(\omega,t) + \int_{t_0}^{t} K(\omega,t,s,x(\omega,s))ds \qquad (3.1)$$

where $x_0 \in C(\Omega \times I,X)$ and $K \in C(\Omega \times I \times I \times X,X)$.

A function $x: \Omega \times I \to X$ is said to be a global random solution of the random equation (3.1), if $x(\omega,t)$ satisfies the random equation (3.1) and $x \in C(\Omega \times I,X)$.

THEOREM 3.1. *Let* $x_0 \in C(\Omega \times I,X)$ *and* $K \in C(\Omega \times I \times I \times X,X)$, *and suppose the following conditions are satisfied:*

(A_1): *for each* $\omega \in \Omega$ *and for all* $(t,s,x) \in I \times I \times X$, $\|K(\omega,t,s,x)\| \leq M(\omega)$, *where* $M:\Omega \to (0,\infty)$ *is a given function*.

(A_2) *there exist positive integers p,q and a function* $L:\Omega \to (0,\infty)$ *such that for each* $\omega \in \Omega$ *and for all* $(t,s) \in I \times I$ *and* $x(\omega,s)$, $y(\omega,s) \in C(\Omega \times I,X)$, *the following inequality holds:*

$$\|K(\omega,t,s,T^{p-1}(\omega,x(\omega,s))) - K(\omega,t,s,T^{q-1}(\omega,y(\omega,s)))\| \leq \qquad (3.2)$$
$$\leq L(\omega)\, D_{\|\cdot\|}(O_T(x(\omega,s),0,\infty) \cup O_T(y(\omega,s),0,\infty)),$$

where $T^n(\omega,x(\omega,t)) = x_0(\omega,t) + \int_{t_0}^{t} K(\omega,t,s,T^{n-1}(\omega,x(\omega,s)))ds$,

$n = 1,2,\ldots,$ *and* $T^0(\omega,x(\omega,t)) = x(\omega,t)$.
Then the random equation (3.1) has a unique global random solution $x^* \in C(\Omega \times I,X)$ *and for each* $x \in C(\Omega \times I,X)$, *the sequence* $\{T^n(\omega,x(\omega,t))\}_{n\geq 0}$ *converges to* $x^*(\omega,t)$ *under the norm* $\|\cdot\|_I$ *for each* $\omega \in \Omega$.

PROOF. Clearly, for each fix $\omega \in \Omega$, $(C(\Omega \times I,X),\|\cdot\|_I)$ is also a separable Banach space. For fixed $\omega \in \Omega$, we shall introduce a new norm on $C(\Omega \times I,X)$ by

$$\|x(\omega,t)\|_* = \max_{t\in I} e^{-L(\omega)t} \|x(\omega,t)\|.$$

Then we have

$$e^{-L(\omega)(t_0+a)}\|x(\omega,t)\|_I \leq \|x(\omega,t)\|_* \leq \|x(\omega,t)\|_I$$

and hence $\|\cdot\|_*$ and $\|\cdot\|_I$ are an equivalent norm. Now we shall discuss the random equation (3.1). By (A_1) and Lemma 2.4, the random nonlinear integral operator $T(\omega,\cdot)\colon (C(\Omega \times I,X),\|\cdot\|_I) \to (C(\Omega \times I,X), \|\cdot\|_I)$ is continuous and, so, $T(\omega,\cdot)$: $(C(\Omega \times I,X),\|\cdot\|_*) \to (C(\Omega \times I,X),\|\cdot\|_*)$ is also continuous. From assumption (A_2), it follows that for each $\omega \in \Omega$

$$\|T^p(\omega,x(\omega,t)) - T^q(\omega,y(\omega,t))\|_* =$$

$$= \max_{t\in I} e^{-L(\omega)t}\| \int_{t_0}^{t} K(\omega,t,s,T^{p-1}(\omega,x(\omega,s))) -$$

$$- K(\omega,t,s,T^{q-1}(\omega,y(\omega,s)))ds\| \leq$$

$$\leq \max_{t\in I} \int_{t_0}^{t} e^{L(\omega)(s-t)} e^{-L(\omega)s} \|K(\omega,t,s,T^{p-1}(\omega,x(\omega,s))) -$$

$$- K(\omega,t,s,T^{q-1}(\omega,y(\omega,s)))\|ds \leq$$

$$\leq L(\omega) \max_{t\in I} \int_{t_0}^{t} e^{L(\omega)(s-t)}ds \cdot D_{\|\cdot\|_*}(O_T(x(\omega,s),0,\infty) \cup$$

$$\cup\ O_T(y(\omega,s),0,\infty)) \leq (1-e^{-L(\omega)a})D_{\|\cdot\|_*}(O_T(x(\omega,s),0,\infty) \cup$$

$$\cup\ O_T(y(\omega,s),0,\infty)) = \beta(\omega)D_{\|\cdot\|_*}(O_T(x(\omega,s),0,\infty) \cup$$

$$\cup\ O_T(y(\omega,s),0,\infty)),$$

where $\beta(\omega) = (1 - e^{-L(\omega)a}) < 1$ for each $\omega \in \Omega$. Since for each $\omega \in \Omega$ and for all $(t,s,x) \in I \times I \times X$, $\|K(\omega,t,s,x)\| \leq M(\omega)$, it is easy to show that for each $\omega \in \Omega$ and for each $x \in C(\Omega \times I,X)$, $D_{\|\cdot\|_*}(O_T(x(\omega,s),0,\infty) < \infty$. Thus from Lemma 2.5 it follows that there exists a unique function $x^*: \Omega \times I \to X$ such that for each fix $\omega \in \Omega$, $x^*(\omega,t)$ satisfies equation (3.1) and for each $x \in C(\Omega \times I,X)$ the sequence $\{T^n(\omega,x(\omega,t))\}_{n\geq 0}$ converges to $x^*(\omega,t)$ under the norm $\|\cdot\|_*$ and so $\{T^n(\omega,x(\omega,t))\}_{n\geq 0}$ converges to $x^*(\omega,t)$ under the norm $\|\cdot\|_I$. On the other hand, from Lemma 2.4 it follows that $\{T^n(\omega,x(\omega,t))\}_{n\geq 0}$ is a $C(I,X)$-valued measurable function sequence. As the limit of $C(I,X)$-valued measurable function sequence, $x^*(\omega,t)$ is $C(I,X)$-valued measurable. By Lemma 2.1 we have $x^* \in C(\Omega \times I,X)$. Thus $x^*(\omega,t)$ is a unique global random solution of the random nonlinear Volterra integral equation (3.1).

COROLLARY 3.2. *Let* $x_0 \in C(\Omega \times I,X)$ *and* $K \in C(\Omega \times I \times I \times X,X)$, *and suppose the following conditions are satisfied:*

(i) condition (A_1) *of Theorem 3.1 holds,*

(ii) there exists a function $L:\Omega \to (0,\infty)$ *such that for each* $\omega \in \Omega$ *and for all* $(t,s) \in I \times I$ *and* $x,y \in C(\Omega \times I,X)$, *the following inequality holds:*

$$(3.3)\quad \|K(\omega,t,s,x(\omega,s)) - K(\omega,t,s,y(\omega,s))\| \leq$$

$$\leq L(\omega)\cdot\|x(\omega,s) - y(\omega,s)\|.$$

Then for each $x(\omega,t) \in C(\Omega \times I,X)$, *the sequence* $\{T^n(\omega,x(\omega,t)\}_{n\geq 0}$ *converges to a unique global random solution* $x^*(\omega,t)$ *of the random equation (3.1) under the norm* $\|\cdot\|_I$.

PROOF. Clearly, corollary 3.2 is a special case of Theorem 3.1 with $p = q = 1$.

Now we shall consider the Cauchy problem of a random nonlinear differential equation:

$$(3.4)\qquad \begin{cases} \dfrac{dx(\omega,t)}{dt} = f(\omega,t,x(\omega,t)) \\ x(\omega,0) = x_0(\omega) \end{cases}$$

where $x_0:\Omega \to X$ is measurable and $f:\Omega \times I \times X \to X$ is a Carathéodory function.

THEOREM 3.3. *Let* $x_0:\Omega \to X$ *be measurable and* $f:\Omega \times I \times X \to X$ *be a Carathéodory function. Suppose the following conditions are satisfied:*

(A_1') there exists a function $M:\Omega \to (0,\infty)$ *such that for each* $\omega \in \Omega$ *and for all* $(t,x) \in I \times X$, $\|f(\omega,t,x)\| \leq M(\omega)$,

(A_2') there exist positive integers p,q *and a function* $L:\Omega \to (0,\infty)$ *such that for each* $\omega \in \Omega$ *and for all* $t \in I$ *and* $x,y \in C(\Omega \times I,X)$ *the following inequality holds:*

$$\|f(\omega,s,T^{p-1}(\omega,x(\omega,s))) - f(\omega,s,T^{q-1}(\omega,y(\omega,s)))\|$$

$$\leq L(\omega)\, D_{\|\cdot\|}(O_T(x(\omega,s),0,\infty) \cup O_T(y(\omega,s),0,\infty)),$$

where

$$T^n(\omega,x(\omega,t)) = x_0(\omega) + \int_{t_0}^{t} f(\omega,s,T^{n-1}(\omega,x(\omega,s))ds,$$

$n = 1,2,\ldots,$ *and* $T^0(\omega,x(\omega,t)) = x(\omega,t)$.

Then for each $x \in C(\Omega \times I,X)$, *the sequence* $\{T^n(\omega,x(\omega,t))\}_{n\geq 0}$ *converges to a unique global random solution* $x^*(\omega,t)$ *of the random Cauchy problem (3.4) under the norm* $\|\cdot\|$.

PROOF. The problem of solving the random Cauchy problem (3.4) is known to be equivalent to that of solving the following random nonlinear Volterra integral equation:

$$x(\omega,t) = x_0(\omega) + \int_{t_0}^{t} f(\omega,s,x(\omega,s))ds \ . \tag{3.5}$$

From an application of Theorem 3.1 with $x_0(\omega,t) = x_0(\omega)$ and $K(\omega,t,s,x) = f(\omega,s,x)$ for all $t \in I$, it follows that there exists a unique global random solution $x^*(\omega,t)$ of the random nonlinear Volterra integral equation (3.5). Since the random Volterra equation (3.5) is equivalent to the random Cauchy problem (3.4), therefore $x^*(\omega,t)$ is also a unique global random solution of the random Cauchy problem (3.4).

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REZIME

O GLOBALNOM SLUČAJNOM REŠENJU ZA SLUČAJNE INTEGRALNE I DIFERENCIJALNE JEDNAČINE

Dobijeni su rezultati o egzistenciji, jednoznačnosti i aproksimaciji globalnog slučajnog rešenja za slučajne nelinearne integralne i diferencijalne jednačine, metodom različitom od onih u [1],[2],[3],[4],[5],[6],[7],[8],[9],[10].