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ON AN ERROR ESTIMATION FOR QUINTIC SPLINE SOLUTIONS OF BOUNDARY VALUE PROBLEMS

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ABSTRACT

In [4] a quintic spline difference scheme has been formed for two point boundary value problems involving second order differential equations lacking the first derivative. It has been shown that such a scheme has a 4-th order of convergence. In this paper, it is proved that the scheme [4] has a 5-th order of convergence.

We shall consider the problem

$$(1)\qquad y''(x) = F(x)y(x) + g(x)\ ,\quad y(a) - A_1 = y(b) - A_2 = 0$$

where $F(x)$ and $g(x)$ are smooth enough functions with $F(x) \geq 0$ on $[a,b]$, and a,b,A_1,A_2 are arbitrary real finite constants.

The quintic spline difference scheme [4] for problem (1) has the form

$$(2)\qquad Au = d$$

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where $u = \{u_i\}$, $d = \{d_i\}$ are n-dimensional column vectors and $A = \{a_{ij}\}$ is a $(n \times n)$ matrix such that

$$a_{12} = -2 + 7h^2F_2/6, \quad a_{n,n-1} = -2 + 7h^2F_{n-1}/6$$

$$a_{13} = -1 + h^2F_3/12, \quad a_{n,n-2} = -1 + h^2F_{n-2}/12$$

$$a_{ij} = 7 + 41h^2F_i/12, \quad i = j = 1,n$$

$$a_{ij} = 6 + 33h^2F_i/10, \quad i = j = 2(1)\overline{n-1}$$

$$a_{ij} = -2 + 13h^2F_i/10, \quad |i-j| = 1, \quad i,j = 2(1)\overline{n-1}$$

$$a_{ij} = -1 + h^2F_i/20, \quad |i-j| = 2, \quad i,j = 3(1)\overline{n-2}$$

$$a_{ij} = 0, \quad |i-j| > 2$$

$$h = (b-a)/(n+1), \; n > 3, \quad F_i = F(x_i), \; x_i = a + ih, \; i = 0(1)\overline{n+1}$$

$$d_1 = (4 - h^2F_o/3)A_1 - h^2(4g_o + 41g_1 + 14g_2 + g_3)/12$$

$$d_2 = (1 - h^2F_o/20)A_1 - h^2(g_o + 26g_1 + 66g_2 + 26g_3 + g_4)/20$$

$$d_i = -h^2(g_{i-2} + 26g_{i-1} + 66g_i + 26g_{i+1} + g_{i+2})/20, \; i = 3(1)\overline{n-2}$$

$$d_{n-1} = (1 - h^2F_{n+1}/20)A_2 - h^2(g_{n-3} + 26g_{n-2} + 66g_{n-1} + 26g_n + g_{n+1})/20$$

$$d_n = (4 - h^2F_{n+1}/3)A_2 - h^2(g_{n-2} + 14g_{n-1} + 41g_n + 4g_{n+1})/12 \; .$$

Let $z_i = y(x_i) - u_i$ be a difference between the exact and the approximate solution at point x_i. Then the error equation (according to [4]) has the form

$$Az = t \tag{3}$$

where the vector $t = \{t_i\}$ has the components

$$t_i = \begin{cases} -h^6y^{(6)}(w_1)/48, & x_o < w_1 < x_3, \quad i = 1 \\ h^6y^{(6)}(w_i)/120, & x_{i-2} < w_i < x_{i+2}, \quad i = 2(1)\overline{n-1} \\ -h^6y^{(6)}(w_n)/48, & x_{n-2} < w_n < x_{n+1}, \quad i = n \end{cases} \; . \tag{4}$$

Let A_o be a five band matrix obtained from A by setting each $F_i = 0$

$$A_o = \begin{bmatrix} 7 & -2 & -1 & & & & \\ -2 & 6 & -2 & -1 & & & \\ -1 & -2 & 6 & -2 & -1 & & \\ & \cdot & \cdot & \cdot & \cdot & & \\ & & -1 & -2 & 6 & -2 & -1 \\ & & & -1 & -2 & 6 & -2 \\ & & & & -1 & -2 & 7 \end{bmatrix} .$$

In [4] it has been shown that:

a) $A^{-1} < A_o^{-1}$ if $13h^2F_i < 20$, $i = 1(1)n$

b) $A_o = PQ$

where $P = \{p_{ij}\}$, $Q = \{q_{ij}\}$ are tridiagonal matrices with

$p_{ii} = 2$, $p_{ij} = -1$, $|i-j| = 1$, $q_{ii} = 4$, $q_{ij} = 1$, $|i-j| = 1$.

c) $A_o^{-1} = (P^{-1} + Q^{-1})/6$.

We shall consider two systems:

$$A_o\bar{z} = \omega, \quad \omega = \{\omega_i\}, \quad \omega_i = \max_{i<j<n} |z_j|, \quad i = 1(1)n, \quad \text{and}$$

$$Az = t \quad .$$

From [1] (p.129) and [2] (p.269), it follows that

(5) $$|z_i| < \bar{z}_i$$

$$\bar{z} = A_o^{-1}\omega = \frac{1}{6}(P^{-1} + Q^{-1})\omega = \frac{1}{6}(\phi + \psi)$$

$$\phi = P^{-1}\omega, \quad \psi = Q^{-1}\omega \quad .$$

Since $\phi_i = \frac{i(n+1-i)}{2}\cdot\omega_i$, $i = 1(1)n$, and

$$\max_{1<i<n} |\psi_i| < \|Q^{-1}\|_\infty \cdot \omega_i < \frac{1}{2}\omega_i = 0(h^6)$$

we have

$$\bar{z}_i < \frac{1}{6}(\phi_i + \frac{1}{2}\omega_i) \tag{6}$$

From (4) and (5) we have

$$\begin{cases} \bar{z}_i < \frac{1}{12}\omega_i(n+1) = 0(h^5), & i = 1,n \\ \bar{z}_i < \frac{1}{2}\omega_i(2n-1) = 0(h^5), & i = 2,\overline{n-1} \end{cases} \tag{7}$$

From the first equation of system (3) with respect to (5) and (7), we have $z_i = 0(h^5)$, $i = 1,2,3$, because $a_{ij} = 0(1)$, $i,j = 1(1)n$.

Since system (3) has a unique solution, from the other equations of system (3), we can conclude (by induction), that $z_i = 0(h^5)$, $i = 4(1)n$. The following theorem is valid.

THEOREM. *Let boundary value problem (1) with* $F(x) > 0$ *have a solution* $y(x) \in C^6[a,b]$. *Let* $\{u_j\}$ *be the approximate solution of (1) obtained by using (2) where* h *is chosen such that*

$$13h^2F(x_i) < 20, \qquad i = 1(1)n \quad .$$

There is, then, a constant M *independent on* h, *such that*

$$|y(x_j) - u_j| < Mh^5 , \qquad i = 1(1)n.$$

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REZIME

O OCENI GREŠKE ZA SPLAJN-PETOG STEPENA

REŠENJA KONTURNIH PROBLEMA

U [4] je primenom splajn funkcije petog stepena konstruisana diferencna šema za rešavanje konturnog problema (1). Dokazano je da šema ima četvrti red tačnosti. Ovde je pokazano da navedena šema ima peti stepen tačnosti.