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A NOTE ON AN ITERATIVE PROCESS

Dragoslav Herceg, *Nenad Petrović***

Prirodno-matematički fakultet. Institut za matematiku 21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija

**Viša tehnička škola, 21000 Novi Sad, ul. Školska br.1, Jugoslavija*

ABSTRACT

In this paper we consider the numerical solution of the equation $f(x) = 0$ on the interval $D = [a,b] \subset R$, for a real-valued function f, by the iterative process (1) from [1]. For this method we give some sufficient conditions for the convergence, and also prove the stopping inequality for $n = 1,2,\ldots$.

INTRODUCTION

We propose to consider the solution of $f(x) = 0$, $x \in D = [a,b] \subset R$, by the iteration

(1) $$x_{n+1} = F(x_n, c), \quad n = 0,1,\ldots,$$

where

(2) $$F(x,c) = x - \frac{f(x)}{2(f(x) - f(c))}\left(\frac{f(x) - 2f(c)}{f'(x)} + \frac{f(x)}{f'(c)}\right),$$

with a suitably chosen $x_o, c \in D$. Method (1) is considered in

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[1] with $x_o = a$, $c = b$ and $x_o = b$, $c = a$, and some sufficient conditions for its converges are given there. We shall give some new sufficient conditions for the convergence of iteration (1) and for the stopping inequality

(3) $$|\alpha - x_{n+1}| \leq |x_{n+1} - x_n|, \quad n = 0,1,\ldots,$$

where $\alpha \in D$ is the solution of $f(x) = 0$.

We shall consider the iterative process (1) under the assumption that the equation $x = F(x,c)$ has a root which coincides with those of $f(x) = 0$ in the interval D, and no others. First, we shall give some notations and assumptions.

Let the function f satisfy the following conditions:

(F) $$f(a) < 0 < f(b), \qquad f'(x) > 0, \qquad x \in D.$$

These conditions imply that $f \in C(D)$ has one and only one root $\alpha \in (a,b)$. Let

$$D^- = [a,\alpha), \quad D_o^- = [a,\alpha], \quad D^+ = (\alpha,b], \quad D_o^+ = [\alpha,b],$$

and let $G(S,k)$, $k \geq 2$, be the class of functions

$$G(S,k) = \{f;\ f : S \subset R \to R,\ f \text{ is } k \text{ times differentiable on } S,\ f^{(k)}(x) \geq 0,\ x \in S\}.$$

For the iterative process $x_{n+1} = g(x_n)$, $n = 0,1,\ldots,$ the next theorem is well known. This theorem shall be used in the proof of the convergence of (1).

THEOREM 1. *Let the equation* $x = g(x)$ *have on* D *the unique solution* α *and let* $g \in G(D_o^+,1)$. *If* $x_o \in D^+$ *satisfies* $x_o > g(x_o)$, *then the sequence* $x_o, x_1, \ldots,$ *generated by* $x_{n+1} = g(x_n)$, $n = 0,1,\ldots,$ *converges to* α *and* $\alpha \leq x_{n+1} \leq x_n$, $n = 0,1,\ldots,$.

If we replace, in this theorem, D_o^+ by D_o^-, D^+ by D^- and $x_o > g(x_o)$ by $x_o < g(x_o)$, then the sequence x_o, $x_1, \ldots,$ generated by $x_{n+1} = g(x_n)$, converges to α and

$x_n < x_{n+1} < \alpha$, $n = 0,1,\ldots$. Theorem 1 is also true if we replace D_o^+ by A_o^+ and D^+ by A^+, where $A_o^+ = [\alpha,c)$ or $A_o^+ = [\alpha,c]$, for some $\alpha < c < b$, and $A^+ = A_o^+\setminus\{\alpha\}$.

1. ON THE ITERATIVE PROCESSES (1)

In this section we shall consider (1) under the assumption (F) and $-f \in G(D_o^+,3)$ or $-f \in G(D_o^-,3)$, and with a different choosing of x_o and c.

From (2), a direct calculation reveals that

$$(4) \qquad F'(x,c) = f(x)\,\frac{f(x) - 2f(c)}{2(f(x) - f(c))}\left(\frac{f''(x)}{f'(x))^2} - \frac{f''(y)}{f'(y)f'(c)}\right) ,$$

where $y \in (\min(x,c), \max(x,c))$. Now, it easy to see that $F(\alpha,c) = \alpha$, $F'(\alpha,c) = 0$, and one can prove that $f''(x)/(f'(x))^2$ is a monotone decreasing function on D_o^+. From (4) follows:

$$(5) \qquad c \in (\alpha,b], \quad f \in G([\alpha,c),2),$$

$$-f \in G([\alpha,c),3) \Rightarrow F \in G([\alpha,c),1),$$

$$(6) \qquad c \in [a,\alpha), \quad -f \in G((c,\alpha),2),$$

$$-f \in G((c,\alpha],3) \Rightarrow F \in G((c,\alpha],1) .$$

If $f''(x)$ has only one zero $\beta \in D$ (from $-f \in G(D,3)$ it follows that f'' has at most one root in D), then

$$(7) \quad c \in (\alpha,b], \quad \beta < \alpha, \quad -f \in G([\alpha,c),3) \Rightarrow F \in G([\alpha,c),1),$$

$$(8) \quad \alpha < \beta, \quad c \in (\alpha,\beta), \quad -f \in G([\alpha,c),3) \Rightarrow F \in G([\alpha,c),1).$$

The choosing of the constant c in (5), (6), (7) is simple, i.e. $c = b$ or $c = a$.

Applying Theorem 1, we have the next theorems.

THEOREM 2. *Let* f *satisfy* (F) *and let* $f \in G(D_o^+,2)$, $-f \in G(D_o^+,3)$, $c \in (\alpha,b]$ *and* $x_o \in (\alpha,c)$. *Then the iterative* *process (1) converges to the unique solution* α *of* $f(x) = 0$, *it is of second order,* $\alpha < x_{n+1} < x_n$, $n = 0,1,\ldots$.

PROOF: The second order of convergence of the iteration (1) follows from $F(\alpha,c) = \alpha$, $F'(\alpha,c) = 0$. From (5) we have $F \in G([\alpha,c),1)$ and from (2) for $x \in (\alpha,c)$ follows $F(x,c) < x$, since

$$f(x)-f(c)<0,\ \frac{f(x)-2f(c)}{f'(x)}+\frac{f(x)}{f'(c)}<\frac{f(x)-2f(c)+f(x)}{f'(x)}<0\ .$$

One can now apply Theorem 1.

THEOREM 3. *Let* f *satisfy* (F) *and let* $-f \in G(D_o^-,2)$, $-f \in G(D_o^-,3)$, $c \in [a,\alpha)$ *and* $x_o \in (c,\alpha)$. *Then the iterative process (1) converges to the unique solution* α *of* $f(x) = 0$, *it is of second order, and* $x_n < x_{n+1} < \alpha$, $n = 0,1,\ldots$.

PROOF: We need prove only that $F(x,c) > x$, for $x \in (c,\alpha)$. From (2) and (F) follows

$$f(x)-f(c)>0,\quad \frac{f(x)-2f(c)}{f'(x)}+\frac{f(x)}{f'(c)}>\frac{f(x)-2f(c)+f(x)}{f'(c)}>0\ ,$$

such that $F(x,c) > x$.

THEOREM 4. *Let* f *satisfy* (F) *and let* $-f \in G(D_o^+,3)$. *If* f'' *has only one zero* $\beta \in D$ *and if*

$$c \in \begin{cases} (\alpha,b] & \text{if } \beta < \alpha, \\ (\alpha,\beta) & \text{if } \alpha < \beta, \end{cases} \qquad x_o \in (\alpha,c),$$

then iteration (1) converges to the unique solution α *of* $f(x) = 0$, *it is of second order, and* $\alpha < x_{n+1} < x_n$, $n = 0,1,\ldots$.

PROOF: Using (7), (8), we see that under our assumption $F \in G([\alpha,c),1)$ and $x_1 < x_o$. Now we can apply Theorem 1.

The iterative method (1) was studied in [1] only under the next assumption on f : f satisfies (F), $-f \in G(D,2)$, $h'''(y) < 0$, $y \in [f(a),f(b)]$, where h is the inverse of f.

Using the result from [2], we have

THEOREM 5. *Let* f *satisfy* (F), $f \in G(D,2)$, $-f \in G(D_o^+,3)$. *Let* $c \in (\alpha,b]$, $x_o \in (\alpha,c)$ *and let* $f'(b) < 2f'(a)$. *Then the stopping inequality (3) is valid for all* $n = 0,1,\ldots,$ *where* $x_o, x_1,\ldots$ *is generated by (1).*

PROOF: From (2) we have

$$F(x,c) = x - \frac{f(x)}{f'(x)}\, g(x) \;,$$

where

$$g(x) = 1 - \frac{f(x)}{2(f(x) - f(c))}\left(1 - \frac{f'(x)}{f'(c)}\right) .$$

Since $f'(c) \geq f'(x)$, $f(c) > f(x)$ for $x \in (\alpha,c) \subset D_o^+$, we have $g(x) \geq 1$, $x \in (\alpha,c)$. From (5) we have $F'(x,c) \geq 0$, $x \in (\alpha,c)$. Now we can apply Theorem 2 from [2], with $C_o^+ = [\alpha,c)$, $C^+ = (\alpha,c)$.

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REZIME

BELEŠKA O JEDNOM ITERATIVNOM PROCESU

U radu se posmatra rešavanje jednačine $f(x) = 0$ u intervalu $D = [a,b]$, pri čemu je f realna funkcija realne promenljive, iterativnim postupkom (1) sa funkcijom koraka (2). Pri tom se posmatraju razni izbori konstante c i početne iteracije x_o. Dati su neki dovoljni uslovi za konvergenciju postupka (1), koji su različiti od uslova datih u [1]. Takodje je dokazana nejednačina zaustavljanja (3) za taj postupak.