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Serija za matematiku, 14,2 (1984)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 14, 2 (1984)

PERMUTABLE n-GROUPOIDS

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ABSTRACT

In this paper i-permutable n-groupoids are defined and considered. An n-groupoid (G,f) is called i-permutable iff $f(x_1,\ldots,x_{i-1},f(x_1\ldots,x_n),x_i,\ldots,x_{n-1}) = x_n$ for all $x_1,\ldots,x_n \in G$ and fixed $i \in \{1,\ldots,n\}$. i-permutable n-groupoids represent a generalization of several classes of binary and n-ary groupoids: semisymmetric groupoids, groupoids satisfying Sade's left "key's" law and cyclic n-groupoids. Examples of i-permutable n-groupoids are given and some properties of such n-groupoids described. i-permutable n-groupoids satisfyng some commutativity and associativity conditions are studied. Several conditions for an i-permutable n-groupoid to be an n-group are determined.

1. INTRODUCTION

As it is well known, a binary groupoid is called semi-

AMS Mathematics Subject Classification (1980): 20N15.
Key words and phrases: n-groupoid, n-quasigroup, n-semigroup, n-group, commutativity, associativity.

symmetric iff it satisfies the identity $(xy)x = y$. Every semisymmetric groupoid is necessarily a quasigroup. Semisymmetric quasigroups were investigated in various directions - algebraic and combinatorial. Idempotent semisymmetric quasigroups are equivalent to Mendelsohn triple systems which represent an ordered analogue of Steiner triple systems and some other classes of such quasigroups are related to geometry of plane curves.

In [10] so called cyclic n-quasigroups, which are a generalization of semisymmetric quasigroups, were considered. An n-groupoid (G,f) is cyclic iff the identity $f(f(x_1,\ldots,x_n), x_1,\ldots,x_{n-1}) = x_n$ holds and every such n-groupoid is necessarily an n-quasigroup. Some questions concerning cyclic n-quasigroups and their combinatorial applications were considered in [11], [12], [14].

Now we shall define and consider a class of n-groupoids which represent a generalization of semisymmetric groupoids, groupoids satisfying Sade's left "key's" law and cyclic n-quasigroups (the identity $x(xy) = y$ is called Sade's left "key's" law [1]).

2. NOTATION AND DEFINITIONS

We shall use the following abreviated notation: $f(x_1,\ldots,x_k,x_{k+1},\ldots,x_{k+s},x_{k+s+1},\ldots,x_n) = f(x_1^k, \overset{(s)}{x}, x_{k+s+1}^n)$, whenever $x_{k+1} = x_{k+2} = \ldots = x_{k+s} = x$ (x_i^j is the empty symbol for $i > j$ and for $i > n$, also $\overset{(0)}{x}$ is the empty symbol).

To avoid repetitions we assume throughout the whole text that $n \geqslant 2$.

We say that an n-groupoid (G,f) is k-*solvable*, where $k \in \{1,\ldots,n\} = N_n$ is fixed, iff the equation

$$(1) \qquad f(a_1^{k-1}, x, a_{k+1}^n) = b$$

has a solution $x \in G$ for all elements $a_1^n, b \in G$. If the solution is unique, then (G,f) is called an *uniquely* k-*solvable* n-*groupoid*. If this equation has an unique solution for every $k \in N_n$, then (G,f) is called an n-*quasigroup*.

An n-groupoid (G,f) is (i,j)-*associative*, where $1 \leq i < j \leq n$, iff

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $x_1^{2n-1} \in G$, i.e. iff $f \overset{i}{+} f = f \overset{j}{+} f$.

An n-groupoid (G,f) is *associative* (i.e. it is an *n-semigroup*) iff it is (i,j)-associative for every pair (i,j), $i,j \in N_n$. Note that for the associativity of (G,f) it is sufficient to postulate the $(1,j)$-associativity for all $j \in \{2,3,\ldots,n\} = N_{2,n}$.

An associative n-quasigroup is an *n-group*. One can prove (see [8] p. 213) that an n-semigroup is an n-group iff it is k-solvable for $k = 1$ and $k = n$ or for some k other than 1 and n. On the other hand, Sokolov [9] proved (but this proof is not complete, cf. [4]) that an n-quasigroup is an n-group iff it is $(i,i+1)$-associative for some $i \in N_{n-1}$. A similar characterization of n-groups is given in [2] and [4].

An n-groupoid (G,f) is (i,j)-*commutative* iff

$$f(x_1^{i-1}, x_i, x_{i+1}^{j-1}, x_j, x_{j+1}^{n}) = f(x_1^{i-1}, x_j, x_{i+1}^{j-1}, x_i, x_{j+1}^{n})$$

for all $x_1^n \in G$ and some fixed $1 \leq i < j \leq n$. If (G,f) is (i,j)-commutative for every pair (i,j), $i,j \in N_n$, then it is *commutative*. Since the symmetric group of degree n S_n is generated by the set of all transpositions $(1,j)$, where $j \in N_{2,n}$, then for the commutativity of (G,f) it suffices only to postulate the $(1,j)$-commutativity for all $j \in N_{2,n}$. If (G,f) is an n-group, then it is commutative iff it is $(i,i+1)$-commutative for some $i \in N_{n-1}$ (see [2]).

If (G,f) is a k-groupoid and $n = t(k - 1) + 1$ for some natural t, then an groupoid $(G,f_{(t)})$ is defined by

$$f_{(t)}(x_1^n) = f(f(\dots f(f(x_1^n),x_{k+1}^{2k-1}),\dots),x_{(t-1)(k-1)+2}^n).$$

3. PERMUTABLE n-GROUPOIDS

DEFINITION 1. *An* n*-groupoid* (G,f) *is called* i*-permutable iff*

$$f(x^{i-1},\ f(x^n),\ x_i^{n-1}) = x_n$$

for all $x_1^n \in G$ *and fixed* $i \in N_n$.

This definition can be also given in another from. The next definition is equivalent to Definition 1.

DEFINITION 1′. *An* n*-groupoid* (G,f) *is called* i*-permutable* ($i \in N_n$) *iff for all* $x_1^{n+1} \in G$

$$f(x_1^n) = x_{n+1} \implies f(x_1^{i-1},\ x_{n+1},\ x_i^{n-1}) = x_n.$$

Using the implication from the preceding definition it is easy to obtain the following definition equivalent to the preceding ones.

DEFINITION 1″. *An* n*-groupoid* (G,f) *is* i*-permutable* ($i \in N_n$) *iff for every* $j \in \{i, i+1,\dots, n\} = N_{i,n}$ *and all* $x_1^{n+1} \in G$

$$f(x_1^n) = x_{n+1} \iff f(x_1^{i-1},\ x_{j+1}^{n+1},\ x_i^{j-1}) = x_j.$$

Observe that in an i-permutable n-groupoid (G,f), for all $j \in N_{i,n}$ and a_1^{j-1}, $a_{j+1}^{n+1} \in G$, the equality

$$f(a_1^{j-1},\ x,\ a_{j+1}^n) = a_{n+1} \tag{2}$$

is (by Definition 1″) equivalent to

$$f(a_1^{j-1}, a_{j+1}^{n+1}) = x.$$

This means that the equation (2) has a solution $x \in G$ for all $j \in N_{i,n}$ and that this solution is uniquely determined. Hence the following theorem is true:

THEOREM 1. *Every* i-*permutable* n-*groupoid is uniquely* j-*solvable for all* $j \in N_{i,n}$.

COROLLARY 1. *([10]) Every* 1-*permutable* n-*groupoid is an* n-*quasigroup.*

Note that 1-permutable n-groupoids are cyclic n-quasigroups introduced in [10]. 1-permutable groupoids (binary) are semisymmetric quasigroups and 2-permutable groupoids satisfy Sade's left "key's" law.

Applying the above theorem to Lemma 2 from [3], we obtain:

COROLLARY 2. *Every* i-*permutable* ($i \in N_{n-1}$) n-*semigroup* (G,f) *is cancellative, i.e.*

$$f(a_1^{k-1}, x, a_{k+1}^n) = f(a_1^{k-1}, y, a_{k+1}^n) \Rightarrow x = y$$

for all $x,y,a_1^n \in G$ *and every* $k \in N_n$.

Now we give some examples.

1. Let (G,+) be an arbitrary (binary) Abelian group, c an arbitrary element from G and let φ be an automorphism of the group (G,+) such that $\varphi c = -c$, $\varphi^{n-i+1} x = -x$ if n-i is even, or $\varphi^{n-i+1} x = x$ if n-i is odd. Then by the formula

$$(3) \qquad f(x_1^n) = \varphi x_{i+1} - \varphi^2 x_{i+2} + \varphi^3 x_{i+3} - \dots + (-1)^{n-i+1} \varphi^{n-i} x_n + c$$

an (i+1)-permutable ($0 \leq i \leq n-1$) n-groupoid (G,f) is defined.

If $(R,+,\cdot)$ is an arbitrary associative ring with unity and $b \in R$ is an invertible element, then the mapping $\varphi: x \mapsto bx$ is an automorphism of the additive group of the ring. If b is also such that $b^{n-i+1} = (-1)^{n-i+1}$, then formula (3) with given φ and $c = 0$ defines an $(i+1)$-permutable $(0 \leq i \leq n-1)$ n-groupoid (R,f).

2. Let $(G,+)$ be a Boolean group, $c \in G$. Then (G,f), where

$$f(x_1^n) = x_i + x_{i+1} + \ldots + x_n + c,$$

is an n-groupoid which is j-permutable for every $j \in N_{i,n}$. If $i = 1$, then (G,f) is a commutative n-group. This n-group is j-permutable for every $j \in N_n$.

3. Let $(G,+)$ be a Boolean group and let

$$g(x_1^n) = x_1 + x_2 + \ldots + x_i + x_j + x_{j+1} + \ldots$$
$$\ldots + x_n + c,$$

where $1 \leq i < j \leq n$. Then an n-groupoid (G,g) is k-permutable for every $k \in N_{j,n}$. But if $i < j-1$ and $|G| > 1$ it is not k-permutable for any $k < j$.

4. A set G such that $|G| > 1$, with the operation $g(x_1^n) = x_n$ is an n-semigroup which is i-permutable iff $i = n$.

5. If (G,g) is an i-permutable k-groupoid $(1 \leq i \leq k \leq n)$, then an n-groupoid (G,f) with $f(x_1^n) = g(x_{n-k+1}^n)$ is $(n-k+i)$-permutable.

By direct computations we obtain the following lemmas.

LEMMA 1. *If an k-groupoid* (G,f) *is* i-*permutable for some* $i \in N_{2,k}$, *then an* $(t(k-1)) + 1)$-*groupoid* $(G,f_{(t)})$ *is*

$((t-1)(k-1)+i)$*-permutable.*

LEMMA 2. *Let* (G,f) *be an* $(i-1,i)$*-comutative* n*-groupoid, where* $i \in N_{2,n}$. *Then* (G,f) *is* i*-permutable iff it is* $(i-1)$*-permutable.*

PROPOSITION 1. *If an* i*-permutable* n*-semigroup is* $(i-1, i)$ *or* $(i,i+1)$*-commutative, then it is commutative.*

PROOF. Let (G,f) be an i-permutable ($i \in N_{2,n}$) n-semigroup. If it is $(i-1,i)$-commutative, then by Lemma 2 it is also $(i-1)$-permutable. Hence we have the following identity

$$y = f(x_1^{i-1}, f(x_1^{n-1},y), x_i^{n-1}) = f(x_1^{i-2}, f(x_{i-1},x_1^{n-1}), y, x_i^{n-1}),$$

which implies that in G there exist elements $z_1, z_2, \ldots, z_{n-1}$ such that

$$y = f(z_1^{i-1}, y, z_i^{n-1}) = f(z_1^{i-2}, y, z_{i-1}^{n-1})$$

for all $y \in G$. Thus by Theorem 5 from [2] we obtain the commutativity of (G,f).

The proof of the second case is analogous.

Since every i-permutable n-groupoid (G,f) is surjective, i.e. for every $y \in G$ there exist elements $x_1^n \in G$ such that $y = f(x_1^n)$, then Corollary 14 from [2] gives the next characterization of the commutativity of i-permutable n-semigroups.

COROLLARY 3. *An* i*-permutable* n*-semigroup* (G,f) *is commutative iff there exists* $k \in N$ *such that* $(G,f_{(k)})$ *is commutative.*

Now we characterize the (i,j)-associative n-groupoids.

THEOREM 2. *Let* (G,f) *be an* i*-permutable* n*-groupoid where* $1 \leq i \leq j < k < n$. *Then* (G,f) *is* (j,k)*-associative iff it is* $(j+1,k+1)$*-associative.*

PROOF. Let (G,f) be an i-permutable (j,k)-associative n-groupoid, $1 \leq i \leq j < k < n$. Then for all $x_1^{2n} \in G$

$$(4) \qquad f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}) = x_{2n}$$

$$\Longleftrightarrow \quad f(x_1^{k-1}, f(x_k^{k+n-1}), x_{k+n}^{2n-1}) = x_{2n}.$$

By the i-permutability of (G,f) we have

$$f(x_1^{i-1}, x_{2n}, x_i^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-2}) = x_{2n-1}$$

$$\Longleftrightarrow \quad f(x_1^{i-1}, x_{2n}, x_i^{k-1}, f(x_k^{k+n-1}), x_{k+n}^{2n-2}) = x_{2n-1},$$

which means that for all $x_1^{2n-2}, x_{2n} \in G$

$$(5) \qquad f(x_1^{i-1}, x_{2n}, x_i^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-2}) =$$

$$= f(x_1^{i-1}, x_{2n}, x_i^{k-1}, f(x_k^{k+n-1}), x_{k+n}^{2n-2}),$$

i.e. (G,f) is (j+1, k+1)-associative. It is clear that starting from (5) we can get (4), hence (G,f) is (j,k)-associative iff it is (j+1, k+1)-associative.

COROLLARY 4. *From the preceding theorem it follows that an* i-*permutable* n-*groupoid* (G,f), *where* $1 \leq i < j < k \leq n$, *is* (j,k)-*associative iff it is* (j-1, k-1)-*associative.*

COROLLARY 5. *Assume that an* n-*groupoid* (G,f) *is* i-*permutable and* (j,k)-*associative, where* $1 \leq i \leq j < k \leq n$. *Then if* $k < n$ (G,f) *is also* (j+s, k+s)-*associative for every* $s \in N_{n-k}$ *and if* $i < j$ *it is* (j-t, k-t)-*associative for every* $t \in N_{j-i}$.

4. PERMUTABLE n-GROUPS

Let (G,f) be an i-permutable n-quasigroup which is (j,k) and (p,m)-associative, where $i \leq j < k < n$ and $i \leq p < m < n$. Since it is also (j+s, k+s) and (p+t , m+t)-associative for every

$s \in N_{n-k}$ and $t \in N_{n-m}$, then if $k-j = m-p\pm1$ it is also $(r,r+1)$-associative for some $r \in N_{n-j}$. Thus (G,f) is an associative n-quasigroup. Hence we obtain:

PROPOSITION 2. *An i-permutable (j,k)-associative n-quasigroup* $(1 \leq i \leq j < k < n)$ *is an n-group iff it is also (p,m)-associative for some pair (p,m) such that* $i \leq p < m < n$ *and* $k-j = m-p\pm1$.

On the other hand Proposition 1 from [4] implies the following proposition.

PROPOSITION 3. *Let (G,f) be an i-permutable n-groupoid.*

(i) *If* $i = 1$, *then (G,f) is an n-group iff it is (j,j+1)-associative for some* $j \in N_{n-1}$.

(ii) *If* $i \in N_{2,n-2}$, *then (G,f) is an n-group iff it is (j,j+1)-associative for some* $j \in N_{i,n-2}$.

(iii) *If* $i = n-1$ *or* $i = n$, *then (G,f) is an n-group iff it is (n-1,n)-associative and the equation (1) has a solution (not necessarily unique) for* $k = 1$.

COROLLARY 6. *Every i-permutable n-semigroup* $(i \in N_{n-1})$ *is an n-group.*

The existence of the solution for $k = 1$ in (iii) can be replaced by so-called Dörnte equation or by some additional assumption on the associativity. Indeed, if (n-1)-permutable n-groupoid (G,f) is (n-2, n-1)-associative, then

$$x_n = f(x_1^{n-2}, f(x_1^n), x_{n-1}) = f(x_1^{n-3}, f(x_{n-2}, x_1^{n-1}), x_n, x_{n-1})$$

for all $x_1^n \in G$, which implies

$$y = f(\overset{(n-3)}{x}, f(\overset{(n)}{x}), y, x)$$

for all $x,y \in G$. This shows that for every $x \in G$ there exists

$\hat{x} = f(\overset{(n)}{x}) \in G$ such that $f(\overset{(n-3)}{x}, \hat{x}, y, x) = y$ for all $y \in G$. Hence by Theorem 2 from [2] we have:

PROPOSITION 4. *An* (n-1)-*permutable* n-*groupoid is an* n-*group iff it is* (n-2, n-1) *and* (n-1, n)-*associative.*

If an n-permutable n-groupoid (G,f) is (n-1,n)-associative, then for every $x \in G$ there exists $\hat{x} \in G$ (for example $\hat{x} = f(\overset{(n)}{x})$) such that $f(\overset{(n-1)}{x}, \hat{x}) = x$ and $f(\overset{(n-2)}{x}, \hat{x}, y) = y$ for all $y \in g$. Hence by Corollary 1 from [2] we obtain:

PROPOSITION 5. *An* n-*permutable* n-*groupoid* (G,f) *is an* n-*group iff it is* (n-1, n)-*associative and for every* $x \in G$ *there exists* $\hat{x} \in G$ *such that* $f(y, \overset{(i-2)}{x}, \hat{x}, \overset{(n-i)}{x}) = y$ *for all* $y \in G$ *and some fixed* $i \in N_{2,n}$.

REMARK 1. In this proposition $\hat{x}$ may be equal to $f(\overset{(n)}{x})$. But by a simple generalization of the proof of Corollary 1 from [2] and Theorem 2 from [4] we can prove that an (n-1, n)-associative n-groupoid (G,f) is an n-group iff for all $x \in G$ there exist $\hat{x}, \tilde{x} \in G$ such that

$$f(\overset{(n-j)}{x}, \tilde{x}, \overset{(j-2)}{x}, y) = f(y, \overset{(i-2)}{x}, \hat{x}, \overset{(n-i)}{x}) = y$$

for all $y \in G$ and some fixed $i,j \in N_{2,n}$.

From Theorem 2 in [2] follows also:

PROPOSITION 6. *If* (G,f) *is an* i-*permutable* n-*groupoid* $(i \in N_{2,n-1})$ *and* $(G, f \overset{i}{+} f)$ *is* (n+i-2,n+i-1) *and* (n+i-1, n+i)-*associative, then* $(G, f \overset{i}{+} f)$ *is an* (2n-1)-*group. Conversely, if* (G,f) *is an* i-*permutable* n-*groupoid* $(i \in N_{2,n-1})$ *which is* (j,j+1)-*associative for some* $j \in N_{n-1}$, *and* $(G, f \overset{i}{+} f)$ *is an* (2n-1)-*group, then* (G,f) *is an* n-*group.*

Since (G,f) is an n-group iff $(G, f_{(2)})$ is an n-group, then in the the case of the (1,i)-associativity Proposition 6 has a specialy simple from.

COROLLARY 7. *An i-permutable* (1,i)*-associative* n*-groupoid* (G,f) ($i \in N_{2,n-1}$) *is an* n*-group iff* $(G,f_{(2)})$ *is* (n+i-2, n+i-1) *and* (n+i-1, n+i)*-associative.*

REMARK 2. In the case of the 1-permutability it suffices to postulate that $(G,f_{(2)})$ is (j,j+1)-associative for some $j \in N_{2n-2}$.

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Received by the editors March 12,1985.

REZIME

PERMUTABILNI n-GRUPOIDI

U ovom radu definisani su i proučavani i-permutabilni n-grupoidi. n-grupoid (G,f) se naziva i-permutabilan ako i samo ako je $f(x_1,\ldots,x_{i-1}, f(x_1,\ldots,x_n), x_i,\ldots,x_{n-1}) = x_n$ za svako $x_1,\ldots, x_n \in G$ i fiksirano $i \in \{1,\ldots,n\}$. i-permutabilni n-grupoidi predstavljaju generalizaciju polusimetričnih grupoida, grupoida koji zadovoljavaju levi Sadov zakon "ključeva" i cikličkih n-grupoida. Navedeni su primeri i-permutabilnih n-grupoida i odredjene neke njihove osobine. Posebno su proučavani i-permutabilni n-grupoidi koji zadovoljavaju zakone komutativnog i asocijativnog tipa. Odredjeni su neki uslovi pod kojima je i-permutabilni n-grupoid n-grupa.