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A CHARACTERIZATION OF HEXAGONAL SYSTEMS

Rade Doroslovački, Ratko Tošić***

**Fakultet tehničkih nauka. Institut za primenjene osnovne discipline, 21000 Novi Sad, V.Vlahovića br.3, Jugoslavija*

***Prirodno-matematički fakultet. Institut za matematiku, 21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija*

ABSTRACT

In this paper we give a characterization of hexagonal systems, which enables us to construct an algorithm for generating all nonisomorphic hexagonal systems with given perimeters.

1. DEFINITIONS AND NOTATIONS

It is known that the Euclidean plane can be tiled with regular congruent hexagons. In this way an infinite hexagonal grid H is obtained. This grid is a plane realization of an infinite planar graph H with vertices of degree 3 and all faces bounded by exactly six edges. Any circuit ζ of the length n in H, in this realization, is represented by a closed broken line C without intersections consisting of n edges. Let H_C be a part of the grid H which belongs either to a circuit C or to its interior. A subgraph H_ζ of the

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graph H whose plane realization is such a part H_C of the grid, H is said to be a hexagonal system. It is clear that two hexagonal systems H_{ζ_1} and H_{ζ_2} are isomorphic graphs iff their planar realizations H_{C_1} and H_{C_2} are congruent geometric figures. So, without loss of generality we can investigate the graphical properties of hexagonal systems using their plane realizations in the grid H. For this reason, we also call H_C a hexagonal system. On the other hand, a hexagonal system H_C is uniquely determined by the corresponding circuit C. This circuit is said to be the perimeter of the hexagonal system.

2. CHARACTERIZATION OF THE HEXAGONAL SYSTEM USING WORDS OVER THE ALPHABET {0,1,2,3,4,5}.

Through any point of the plane of a hexagonal grid, three distinct lines can be drawn, such that each of the hexagonal grid is parallel to exactly one of these three lines. Suppose that these lines are oriented. Denote the unit vectors of the oriented lines by 0,1 and 2, in such a way that $\sphericalangle(0,1) = \frac{\pi}{3}$; $\sphericalangle(0,2) = \frac{2\pi}{3}$. By 3,4,5 we denote the vectors opposite to the vectors 0,1 and 2, respectively. The length of each edge of the grid H is one.

We shall consider the set of vectors $V = \{0,1,2,3,4,5\}$ an alphabet. V^n will be the set of all words of the length n over the alphabet V and $V^* = \bigcup_{n>0} V^n$.

Clearly, to any oriented edge of the grid H, there corresponds, uniquely, a vector from the set V. In this way, a function f is defined, which maps the set of all the finite oriented paths of the grid H in the set V^* (Fig.1).

In Fig.1, the path NM is oriented from the vertex N to the vertex M. The circuit C of the length n determines 2n different closed oriented paths, depending on the choice of a vertex at the start and the orientation of the circuit.

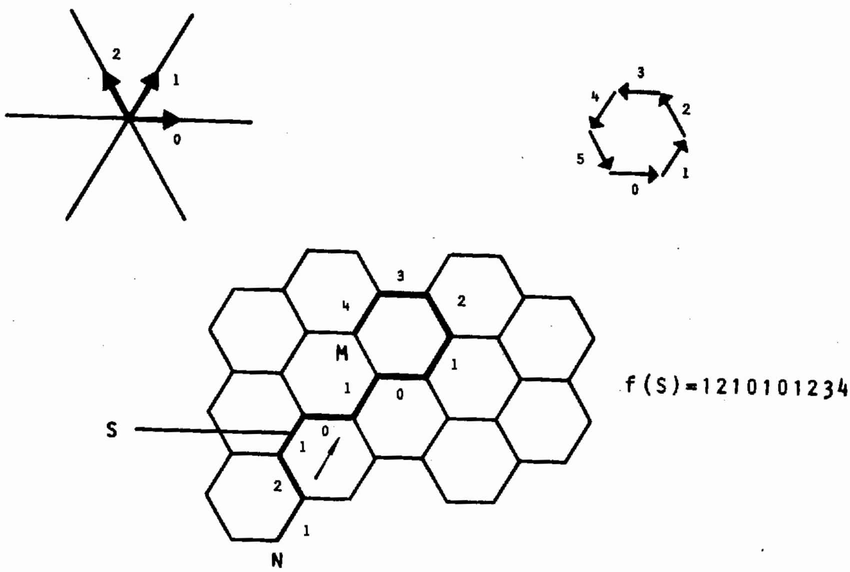

Fig. 1.

Any of these 2n oriented paths determines the same hexagonal system. A function f maps these 2n oriented paths onto 2n different words of the set V^n.

A hexagon h, having at least one edge on the perimeter, C, is said to be a boundary hexagon of H_C.

Let h be a boundary hexagon. By H_C-h we denoted the graph obtained from H_C by first removing all the edges belonging at the same time to C and to h, and afterwords removing all the isolated vertices.

A hexagonal system is said to be kata-hexagonal if all its vertices belong to the perimeter. A kata-hexagonal system is said to be a hexagonal chain, if each of its hexagons is adjacent to at most two other hexagons.

LEMMA. *For each hexagonal system* H_C *with at least two hexagons, there exists at least one boundary hexagon* h *such that* H_C-h *is also a hexagonal system.*

PROOF: Consider the set K_C of all hexagonal chains which are the subgraphs of the hexagonal system H_C. For the hexagonal system H_C, the set K_C must be finite, because H_C contains a finite number of hexagons .:. It follows that there exists $H_m \in K_C$ which is maximal i.e. H_m has the maximum possible number of hexagons. The hexagonal chain H_m contains at least one hexagon h (in fact, exactly two), which is adjacent to exactly one other hexagon (because H_m is finite). Denote the vertices of the hexagon h by A, B, C, D, E, F, and let AF be the common edge of the adjacent hexagons h and h_1 (Fig.2).

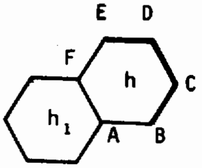

Fig. 2.

The edges BC, CD and DE belong to the perimeter C of H_C (otherwise, H_m would not be according to the maximal). Now we can consider three cases depending on whether some of the edges AB and FE belong to the perimeter or not(Fig.3) .

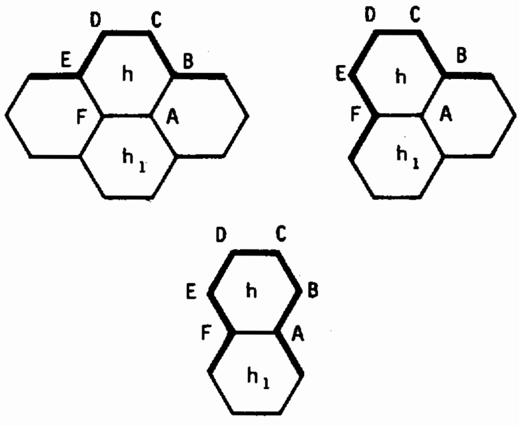

Fig. 3.

Obviously, in each of these three cases, H_C-h is a hexagonal system.

The following theorem gives a sufficient and necessary condition under which the equality

$$p = f(P) ,$$

concerning an arbitrary $p = x_1x_2\dots x_n \in V^n$ $n \geq 6$ and a closed oriented path P without repeated vertices, is valid:

THEOREM. *The equation*

$$p = f(P)$$

is satisfied if and only if each of the following three relationships hold:

(i) $\forall\, k \in N_{n-1}$ $x_{k+1} = x_k \pm 1$, *where we operate with the elements of the set* V *as* V *integers.*

(ii) $\ell_0(p) = \ell_3(p) \wedge \ell_1(p) = \ell_4(p) \wedge \ell_2(p) = \ell_5(p)$

(iii) *For any nonempty subword* r *of* p *different from* p *its holds that*

$$\ell_0(r) \neq \ell_3(r) \vee \ell_1(r) \neq \ell_4(r) \vee \ell_2(r) \neq \ell_5(r)$$

where $\ell_i(q)$ *is the number of appearances in the string* $q \in V^*$ *for* $i \in V$.

PROOF:

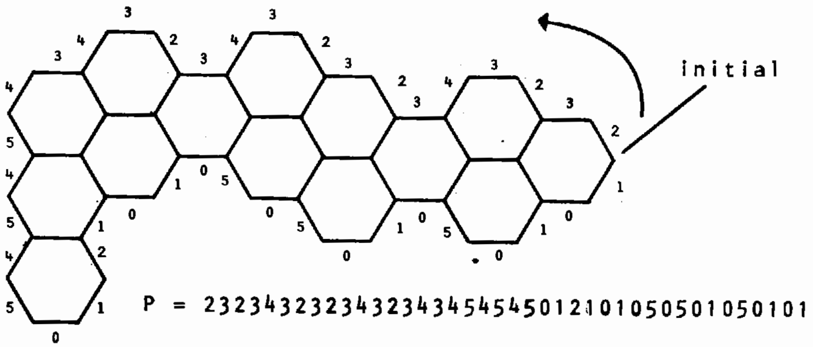

Fig. 4.

NECESSITY:

(i) Follows from the properties of the hexagonal grid (see Fig.1 and Fig.4).

(ii) The proof is by induction on the number m of hexagons of the hexagonal system bounded by the path P_m. For $m = 1$ and $m=2$, the statement can be easily checked. Suppose that (ii) is true for $m = k$ ($k > 2$) i.e. for each path P_k enclosing exactly k hexagons.

Now consider a path P_{k+1} enclosing exactly $k+1$ hexagons. Denote by $H_{C_{k+1}}$ the hexagonal system bounded by P_{k+1}. According to our Lemma, there exists a hexagon h containded in $H_{C_{k+1}}$, such that $H_{C_{k+1}}-h = H_{C_k}$ is a hexagonal system. Let P_k be one of the directed paths which bound H_{C_k}, obtained from P_{k+1} by deleting from P_{k+1} all the edges belonging also to h, and adding all the remaining edges of h, direct in the suitable way. Condition (ii) is true for P_k, according to the induction hypothesis.

By adding the hexagon h, i.e. deleting the edges of P_k belonging to h, and adding the remaining edges of h directed in the suitable way, we again obtain P_{k+1}. It may be shecked that path P_{k+1} satisfies (ii).

(iii) Follows from the fact that there is no repetition of vertices in P.

SUFFICIENCY. Bearing in mind the properties of the grid H, we conclude that for any word satisfying conditions (i), (ii) and (iii), a path P can be constructed such that $p = f(P)$. □

3. ISOMORPHISM OF HEXAGONAL SYSTEMS

Using the Theorem we can determine for any word p over the alphabet $\{0,1,2,3,4,5\}$ whether it represents a hexagonal system or not. But, the same hexagonal system can be represented by different words. Now, we shall define an equivalence relation ρ in the set $V_1^* \subset V^*$ of words satisfying conditions (i), (ii), (iii) of the Theorem, in such a way that all the words from the same equivalence class determine the same hexagonal system up to isomorphism, while the elements from different equivalence classes determine nonisomorphic hexagonal systems.

We use the following notions and definitions:

S - the set of all hexagonal systems

π - the congruence relation in the set S

$\alpha, \beta, \sigma : V_1^* \to V_1^*$ are the following functions

$$\sigma(x_1 x_2 \dots x_n) = \sigma(x_1)\sigma(x_2)\dots\sigma(x_n)$$

where $\sigma : V \to V$ is an arbitrary element of the permutation group generated by σ_a and σ_b where:

$$\sigma_a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix} \qquad \sigma_b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 0 & 5 & 4 \end{pmatrix}$$

$$\alpha(x_1 x_2 \dots x_n) = x_2 x_3 \dots x_n x_1 \qquad \beta(x_1 x_2 \dots x_n) = x_n x_{n-1} \dots x_2 x_1$$

ρ - a binary relation in the set V_1^* defined as follows:

$$\forall p, q \in V_1^* \qquad p \rho q \iff q = (\alpha^k \sigma)(p) \vee q = (\alpha^k \sigma \beta)(p)$$

for some nonnegative integer k.

Functions σ_a and σ_b correspond to a rotation of $\frac{\pi}{3}$ and to an axial symmetry, respectively, therefore the group generated by σ_a and σ_b is isomorphic to the symmetry group of a regular hexagon. The isomorphism between these groups is function f .

Function f in the natural way induces the bijection

$$F : \frac{S}{\pi} \to \frac{V_1^*}{\rho}$$

where $\frac{S}{\pi}$ is the set of all nonisomorphic hexagonal systems, and $\frac{V_1^*}{\rho}$ is the set of all nonequivalent words satisfying conditions (i), (ii) and (iii).

4. SOME COMPUTATIONAL RESULTS

From the given characterization, it follows that all nonisomorphic hexagonal systems with a given perimeter n, can be obtained by generating all the nonequivalent words which satisfy conditions (i), (ii) and (iii). A computational algorithm for generating such words enables us to determine the number of nonisomorphic hexagonal systems H(n) with perimeter n, up to n = 28. The results are given in Table 1.

Table 1.

n	6	8	10	12	14	16	18	20	22	24	26	28
H(n)	1	0	1	1	3	2	12	14	50	97	312	744

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REZIME

JEDNA KARAKTERIZACIJA HEKSAGONALNIH SISTEMA

U ovom radu dajemo jednu karakterizaciju heksagonalnih sistema koja nam omogućava konstrukciju algoritma za generisanje svih neizomorfnih heksagonalnih sistema datog parametra.