ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 14,2 (1984)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 14, 2 (1984)

SOME REMARKS ON p_n - SEQUENCES OF ALGEBRAS

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ABSTRACT

The paper deals with the representation of p_n-sequences in various classes of algebras. A full description of representable sequences, in two-element groupoids, is given. For the sequence $p_n=2^n$, the "smallest" groupoid which represents it is found. The notion of maximality is introduced and some relations with k-valued logics are established.

1. PRELIMINARIES

The definitions of n-ary algebraic operation (or n-ary polynomial), p_n-sequence(cardinalities of essentially n-ary polynomials), and some other notations from Grätzer [5], and Marczewski [7] are adopted here.

The following notation is accepted:

Let $\mathcal{A}=\langle A,F\rangle$ be an algebra. $P^{(n)}(\mathcal{A})$ is the set of all n-ary polynomials over $\mathcal{A}$, $P_{(n)}(\mathcal{A})$ is the set of all different, essentially n-ary polynomials, $p_n(\mathcal{A})$ denotes the cardinality of $P_{(n)}(\mathcal{A})$. (Note: the polynomial $p(x)=x$, for all $x\in A$, is in $P_{(1)}(\mathcal{A})$.)

AMS Mathematics Subject Classification(1980): 08A40.
Key words and phrases: Operations, polynomials, primal algebras.

Let $<p_n(A)>$ denote the sequence $<p_o(A), p_1(A), \dots, p_n(A),$ $,\dots>$. The basic problem in this topic is to study and characterize representable sequences.
Let V be a variety. By $F_V(n)$ we shall denote a free algebra over V, generated by n free generators, Id(V) is the set of all identities satisfied in all algebras from V.
Let τ be a fixed type of algebras, Σ a set of identities in a language of type τ. By Mod (Σ) we shall denote all the algebras of type τ in which all identities from Σ are satisfied. H,S, and P are usual operators on classes of algebras. The following lemma gives the connection between the number of elements in the free algebra, $F_V(n)$ and the polynomial sequences of algebras from V, for the minimal and locally finite varieties.
Here, the lattice of varieties V, of the type τ, is the lattice of all varieties of the type τ.

LEMMA 1. *Let* V *be a locally finite variety, which is minimal in the lattice of varieties, then for all algebras,* $A=<A,F>$, *from* V *with* $|A|>1$, $p_o(A)=p_o$, *then*

$$p_n(A)=p_n=|F_V(n)|-\sum_{k=0}^{n-1}\binom{n}{k}p_k, \text{ for all } n\in N. \tag{1}$$

P r o o f: Let T denote the trivial variety, and let A and B be algebras from V, for which $|A|>1$ and $|B|>1$, therefore, $HSP(A)\neq T$ and $HSP(B)\neq T$. Since $HSP(A)\subseteq V$ and $HSP(B)\subseteq V$, and V is a minimal variety, we have that $HSP(A)=HSP(B)=V$. Using Birkhoff's theorem for equational classes we get: Mod Id(A)=Mod Id(B), which imply Id(A)=Id(B), and $p_n(A)=p_n(B)$.
If $F_V(n)$ is a finite free algebra in V, generated by n free generators, we have: $|F_V(n)|=\sum_{k=0}^{n}\binom{n}{k}p_k(F_V(n))$ [9].

Since $F_V(n)$ is finite (for $n<\omega_o$) in locally finite varieties taking $B=F_V(n)$ we get (1), which completes the proof.

DEFINITION 1. ([1]) *Algebra* $\mathbf{A} = \langle A, \cdot \rangle$ *of type* $\langle 2 \rangle$ *is an implication algebra if the following identities are satisfied:*

A1. $(xy)x = x$ *(contraction)*

A2. $(xy)y = (yx)x$ *(quasi-commutative)*

A3. $x(yz) = y(xz)$ *(exchange)*

All the 2-element algebras have been determined and are explicitly listed by E.Post. J.Berman has redone that classification in [3], where the first few elements of the polynomial sequences of the two-element algebras, are computed. The following proposition gives the full description of the polynomial sequences of the 2-element groupoids.

PROPOSITION 1. *Sequences:* $q_1 = (1,1,0,0,\dots)$, $q_2 = (0,1,0,0,\dots)$, $q_3 = (0,2,0,0,\dots,)$, $q_4 = (0,1,1,1,\dots,)$, $q_5 = (1,1,1,1,\dots)$, $q_6 = (2,2,10,a_3,\dots,a_n,\dots)$, $q_7 = (1,1,3,25, b_4,\dots,b_n,\dots)$, *where*

$$a_n = 2^{2^n} - \sum_{i=0}^{n-1} \binom{n}{i} a_i, \text{ and } b_n = \sum_{i=0}^{n} (-1)^{(i-1)} \binom{n}{i} 2^{2^{(n-i)}} - \sum_{k=0}^{n-1} \binom{n}{k} b_k, \text{ for } n > 1 ;$$

(i) *are representable in the class of 2-element groupoids,* G_2 ;

(ii) *if a sequence is representable in* G_2 *then it is equal to one* q_i, $i = 1,2,\dots,7$.

P r o o f. (i) Let $G = \{0,1\}$. The binary operations, f_i, $i = 1,2,\dots,7$ on G are defined in the following way: $f_1(x,y) = 0$, $f_2(x,y) = x$, $f_3(x,y) = \neg x$, $f_4(x,y) = x \wedge y$, $f_5(x,y) = x \leftrightarrow y$, $f_6(x,y) = \neg(x \wedge y)$, $f_7(x,y) = x \to y$, where $\neg, \wedge, \leftrightarrow, \to$, are usual operations of propositional calculus.

Then algebras $A_i = \langle\{0,1\}, f_i\rangle$ $i = 1,2,\dots,7$, represent the given sequences, q_i, $i = 1,2,\dots,7$, respectively.

It is obvious that for A_1 if $p(x_1,\dots,x_n) \in P^{(n)}(A_1)$, then $p = 0$ or $p = x_i$ for some $i = 1,\dots,n$, i.e. the only nullary polynomial is $p(x) = 0$. $p_1(A_1) = 1$, and the only unary polynomial is $p(x) = x$. $p_i(A_1) = 0$ for $i > 1$, i.e. $q_1 = \langle p_n(A_1)\rangle$.

If $p(x_1,\dots,x_n) = p \in P^{(n)}(A_2)$ then $p = x_i$ for some $i=1,\dots,n$, so $q_2 = \langle p_n(A_2)\rangle$.

If $p(x_1,\dots,x_n) = p \in P^{(n)}(A_3)$ then $p = x_i$ or $p = \neg x_i$, $i=1,\dots,n$, which can be proved by induction on the number of operational symbols, so, $q_3 = \langle p_n(A_3)\rangle$.

For f_4 the following identities are obvious: $f_4(x,x)=x$, $f_4(x,y) = f_4(y,x)$, $f_4(x,f_4(y,z)) = f_4(f_4(x,y),z)$, i.e. A_4 is a semilattice, so $q_4 = \langle p_n(A_4)\rangle$.

If $p \in P^{(1)}(A_5)$ then $p(x) = 1$, or $p(x) = x$, so $p_o(A_5)=1$ and $p_1(A_5) = 1$. The commutative and associative law is satisfied, and $f_5(x,1) = f_5(1,x) = x$. If $p(x_1,\dots,x_n) = \bar{p} \in P^{(n)}(A_5)$, then, $(f_5(x_1,f_5(x_2,f_5(x_3\dots)\dots) = \bar{p}$ so $p_n(A_5) \leq 1$.

The polynomial $\bar{p} \in P_{(n)}(A_5)$, since if $x_k = 1$ for all k, then $\bar{p} = 1$. If $x_i = 0$, and $x_k = 1$ for all $k \neq i$, then $\bar{p} = 0$, i.e. $\bar{p}$ depends in x_i, $i=1,\dots,n$, so, $p_n = 1$. f_6 is a Sheffer-operation in propositional calculus, so A_6 is polynomially equivalent with the 2-element Boolean algebra. Using lemma 1, for the variety of Boolean algebras, we get that $q_6 = \langle p_n(A_6)\rangle$.

A_7 is an implication algebra, and the variety of the implication algebras is minimal and locally finite [1]. Using lemma 1 and the cardinality of the free implication algebra [3] we get that $\langle p_n(A_7)\rangle = q_7$.

(ii) is obvious, for all groupoids from G_2 are polynomially equivalent with one of the groupoids A_i, $i = 1,\dots,7$.

3. For the vector space over GF(3),V, the polynomial sequence is $\langle p_n(V)\rangle = 2^n$, $n \in N$, and the cyclic group of order 3 is polynomially equivalent with it.

The next proposition gives the example of the smallest groupoid which is not a group, but its polynomial sequence is $\langle p_n\rangle = 2^n$, $n \geq 1$ and $p_o = 0$.

PROPOSITION 2. *The sequence* $a = \langle 0, 2^1, 2^2, 2^3, \dots, 2^n, \dots\rangle$ *is representable in the class of the 3-element groupoids, but not in the class of the 2-element groupoids.*

P r o o f. Let $g = (G,+)$ be a groupoid given in table 1. g is commutative, associative, and in it holds that $3x = x$, $(2n)\,x = 2x$, $(2n+1)\,x = x$, $n \in N$.

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{1}$
$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

Table 1.

If $p \in P^{(1)}(g)$, then $p = x$ or $p = 2x$, so $p_o(g) = 0$.

If $p = p(x_1,\dots,x_n) \in P_{(n)}(g)$, then $p = \sum_{i=0}^{n} a_i x_i$, $a_i \in \{1,2\}$, so $p_n(g) \leq 2^n$. It is obvious that p depends on all the variables x_i, $i = 1,\dots,n$.

Let us prove that all polynomials of that form are different. Let $g(x_1,\dots,x_n) = \sum_{i=1}^{n} b_i x_i$ and $a_1 \neq b_1$. Without loosing the generality we can assume that $a_1 = 1$, $b_1 = 2$, so $\bar{1} = p(\bar{1},\bar{2},\dots,\bar{2}) \neq q(\bar{1},\bar{2},\dots,\bar{2}) = \bar{2}$, so the polynomials p and q are different.

Note that according to Proposition 1 this is the smallest groupoid which represents the given sequence.

Some similar sequences, $p_n = 2^n - 1$ and $p_n = 2^{n-1}$ were considered by Płonka and Grätzer [5], [8].

4. PROPOSITION 3. *Let the sequence* $<p_n>$ *be representable in the class of groupoids, and let* $p_2 = 0$. *Then* $p_k = 0$ *for all* $k > 2$.

P r o o f. Let $A = (A,.)$ be a groupoid which represents the given sequence. Since $p_2(A) = 0$ it follows that polynomial $x \cdot y$ does not depend on one variable at least, say y.

Let $p(x_1,\dots,x_n) \in P^{(n)}(A)$, and $n > 2$. We shall prove that $p \notin P_{(n)}(A)$ using the induction on $|p|$, where $|p|$ denotes the lenght of p, i.e. the number of all occurences of all variables in p. If $|p| < n$, then p does not contain some of the variables x_i, $i = 1,\dots,n$, and does not depend on them. If $|p| = n$ then $p(x_1,\dots,x_n) = q(x_1,\dots,x_n) \cdot r(x_1,\dots,x_n)$. As $|q| < n$, q does not depend on one variable at least. Let it be x_1, but then $p(x_1,\dots,x_n)$ does not depend on x_1 either. If this were not so, then there would exist such $a_1', a_1'', a_2, \dots, a_n \in A$ and $q(a_1',a_2,\dots,a_n) \cdot r(a_1',a_2,\dots,a_n) \neq q(a''_1,a_2,\dots \dots,a_n) \cdot r(a_1'',a_2,\dots,a_n)$.

This is in contradiction with the assumption that $x \cdot y$ does not depend on y. Assume that for $|p| < N$, $p \in P_{(n)}(A)$.

Let $|p| = N$. Since $p(x_1,\dots,x_n) = q(x_1,\dots,x_n) \cdot r(x_1, \dots,x_n)$, $|q| < N$, q does not depend on, say x_1. Then p does not depend on x_1 either.

Note that the generalization of this proposition does not hold, i.e. a sequence $(a_o,a_1,\dots,a_{n-2},0,a_n,\dots)$ and $a_k \neq 0$ for some $k > n-1$ may be representable in the class of n-groupoids, i.e. set with an n-ary operation.

It is enough to consider $A = (A,f)$, where $(A,+)$ is an Abelian group of exponent 2, and $f(a,b,c,d) = a+b+c$. Then $p_4(A) = 0$, but $p_5(A) = 1$.

5. Let K_a denote the class of all algebras $\mathcal{A} = (A,F)$ with $|A| = a$. The next proposition gives the upper bound for the numbers $p_n(\mathcal{A})$ in the case for finite algebras $\mathcal{A}$.

PROPOSITION 4. *Let* $\bar{p}_0 = a$ *and for* $k > 0$

$$\bar{p}_k = a^{a^k} - \sum_{i=0}^{k-1} \binom{k}{i} p_i \tag{2}$$

(i) Sequence $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k, \ldots)$ *defined by (2) is representable.*

(ii) For all algebras $\mathcal{B}_1$ *from* K_a, *and all* $k > 0$ $p_k(\mathcal{B}_1) \leq \bar{p}_k$.

P r o o f. (i) Let $\mathcal{A} = \langle\{0,1,\ldots,a-1\},F\rangle$, where $F = \bigcup_{n\in N} F_n$, and F_n is the set of all essentially n-ary functions of A^n into A. The number of all different essentially n-ary functions of A^n into A is determined by (2). Indeed, for $k = 1$, the number of all unary functions on A is a^a, but they are not all essentially unary. There are $\bar{p}_0$ constant functions, so $\bar{p}_1 = a^a - a = a^a - \binom{1}{0}\bar{p}_0$.

Let (2) be satisfied for all $k < n$. Then there are a^{a^n} n-ary functions, but they are not all essentially n-ary. Since there are $\bar{p}_s$ essentially s-ary functions $p(x_1,\ldots,x_s)$, for all $s < n$, then $f(x_1,\ldots,x_n) = p(x_{i_1},\ldots,x_{i_s})$, $i_j \neq i_\ell$, $\{i_\ell,\ldots,i_s\} \subseteq \{1,\ldots,n\}$ is essentially s-ary. For all such subsets of the set $\{1,\ldots,n\}$ there are $\bar{p}_s$ essentially s-ary functions, according to the induction hypothesis.

So, among a^{a^n} n-ary functions there are $\binom{n}{s}\cdot\bar{p}_s$ essentially s-ary functions. Therefore, $\mathcal{A}$ has $\bar{p}_n$ essentially n-ary different fundamentall operations, so $p_n(\mathcal{A}) \geq \bar{p}_n$ for all $n > 0$. However, any polynomial $p(x_1,\ldots,x_n) \in P^{(n)}(\mathcal{A})$ induces some function from A^n into A. Since set A has exactly $\bar{p}_n$ different essentially n-ary functions, we have $p_n(A) \leq \bar{p}_n$. This completes the proof.

(ii) follows from the construction of algebra A, and the fact that every polynomial of $\mathcal{B}_1$ is a function from A^n into A for some n.

DEFINITION 2. *The sequence* $q = \langle q_n \rangle$ *has the maximality property over the class of algebras,* K, *if*

(i) q *is representable in* K,

(ii) *for each algebra,* A, *from* K, $p_n(A) \leq q_n$, *for all* $n \in N$.

Let G_n denote the class of all n-groupoids.

PROPOSITION 5. *Sequence* $\bar{p}_k$, *given in Proposition* 4, *has maximality property in the class*

(i) $K_a \cap G_2$, *for all* a;

(ii) $K_2 \cap G_n$, *for* $n = 2,3,\ldots$

(iii) $K_3 \cap G_n$, *for* $n = 2,3,\ldots$

P r o o f. (i) The operation $f_k(x,y) = \max(x,y) \oplus 1$, ($\oplus$ is addition modulo k), is a Sheffer operation in k-valued logic, and algebra $A = \langle\{0,1,\ldots,a-1\rangle, f_a\}$ represents the given sequence.

(ii),(iii) For the two valued logic and three valued logic there are n-valued Sheffer operations for all $n > 2$.

Notes. For representability of sequence $\bar{p}_k$ in classes $K_a \cap G_n$ for $a > 2$ and $n > 2$ see I.Rosenberg [9].

The notion of maximality of sequences is in close connection with the notion of the primal algebra ([4], § 27.), namely sequence $\bar{p}_k$ has maximality property in a class of algebras with carriers of cardinality a, iff that class has a primal algebra. However, the notion of maximality for sequences is more general than the notion of the primal algebra, or, what is equivalent, with the notion of functional completeness of the set of fundamental operations, F, for an algebra $A = \langle A,F \rangle$.

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Received by the editors December 10,1984.

REZIME

NEKE PRIMEDBE O p_n-NIZOVIMA ALGEBRI

U radu su razmatrani problemi reprezentabilnosti p_n-nizova u raznim klasama algebri. Data je karakterizacija nizova reprezentabilnih u dvoelementnim grupoidima. Za niz $p_n = 2^n$ je data najmanja algebra koja ga reprezentuje. Uveden je pojam maksimalnosti niza i uspostavljena je veza ovog pojma sa k-valentnim logikama.