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THE QUASIASYMPTOTIC BEHAVIOUR OF SOME DISTRIBUTIONS

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ABSTRACT

The quasiasymptotic behaviour at infinity of certain distributions is found explicitely.

I. The aim of this paper is to find explicitly the quasiasymptotic behaviour (q.a.b.) at infinity (see [5]) of certain distributions, namely those which are or which behave (in some sense) at infinity as a regularly varying function (r.v.f.).

Throughout the paper L denotes a locally integrable function (l.i.f.) on $(0,\infty)$ which is slowly varying (s.v.) at infinity. Further, H denotes the Heaviside function, D the distributional derivative.

We shall observe only the q.a.b. at infinity, since it has a non-local property; namely it depends both on the "behaviour" of a distribution at infinity and at finite points.

II. We shall examine first the case when the regular distribution T is defined by a l.i.f.g on $\mathbb{R}$ with

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$\operatorname{supp} T \subset [a,\infty)$ for some $a>0$. We write then $T = H(x-a)g(x)$ and

$$(2.1) \qquad \langle T,\phi\rangle := \int_a^\infty g(x)\phi(x)\,dx, \quad \phi \in S.$$

Let us prove

PROPOSITION 1. *Let* $T = H(x-a)g(x)$ *for* $a>0$ *and the l.i.f.* g *satisfies*

$$\int_a^\infty |g(x)|\,dx < \infty$$

. Then T *has q.a.b. at infinity of order* -1 *related to* $\frac{1}{x}$.

P r o o f. It follows at once from

$$\lim_{k\to\infty} \langle kT(kx),\phi(x)\rangle = \lim_{k\to\infty} \int_a^\infty g(x)\,\phi\left(\frac{x}{k}\right)dx = \langle C\delta,\phi\rangle$$

where

$$C = \int_a^\infty g(x)\,dx.$$

We get a special case of this Proposition if either $g(x) \sim x^\alpha L(x)$ as $x\to\infty$ and $\alpha < -1$ or $g(x) \sim \frac{L(x)}{x}$, $x\to\infty$, but

$$(2.2) \qquad \int_a^\infty \left|\frac{L(x)}{x}\right| dx < \infty.$$

However, if (2.2) is not satisfied, then we have

PROPOSITION 2. *Let* $a>0$ *and* $T(x) = H(x-a)g(x)$, $x \in \mathbb{R}$, *where* g *is a l.i.f. such that* $g(x) \sim \frac{L(x)}{x}$ *as* $x\to\infty$. *If*

$$(2.3) \qquad L^*(x) := \int_a^x \frac{L(x)}{x}\,dx, \quad x \geqslant a$$

diverges to unfinity as $x\to\infty$, *then* T *has q.a.b. of order* -1 *related to* $\frac{L^*(x)}{x}$, $x\to\infty$.

REMARK. By [5], p.86, L^* is also s.v. at infinity (see also Parameswaran [4]).

P r o o f. We observe the function

$$G(x) := (H*T)(x) = \int_a^x g(t)\,dt, \quad x \in \mathbb{R}.$$

Since

$$\lim_{x\to\infty} \frac{G(x)}{L^*(x)} = \lim_{x\to\infty} \frac{g(x)}{\frac{L(x)}{x}} = 1$$

and $\frac{dG}{dx} = T$, the Structural Theorem ([2]) implies the assertion.

At last we have

PROPOSITION 3. *Let* $a > 0$ *and* $T(x) = H(x-a)g(x)$, $x \in \mathbb{R}$, *where* g *is a l.i.f. such that* $g(x) \sim x^{\alpha}L(x)$ *as* $x \to \infty$ *for* $\alpha > -1$. *Then* T *has q.a.b. of order* α *related to* $\rho(x) = x^{\alpha}L(x)$, $x \to \infty$.

P r o o f. It is obvious, since $G(x) := (H*T)(x)$ is a continuous function on $\mathbb{R}$ such that

$$G(x) \sim \frac{x^{\alpha+1}}{\alpha+1} L(x) \quad \text{as} \quad x \to \infty.$$

III. Let us prove (see also [2]):

PROPOSITION 4. *For every* $S \in \mathcal{E}' \cap \mathcal{S}'_+$ *there exists a natural number* n *such that* S *has q.a.b. of order* $-n$, $n \in \mathbb{N}$, *related to* $\frac{1}{x^n}$, $x \to \infty$.

P r o o f. For given $S \in \mathcal{E}'$ there exists $m \in \mathbb{N}_0$ and a continuous function G on $\mathbb{R}$ with $\operatorname{supp} G \subset [0,\infty)$ such that $S = D^m G$ $(D = \frac{d}{dx})$. If $\operatorname{supp} S \subset [0,a]$, $a > 0$, we have that G is equal to some polynomial of order $\leq m-1$ on the interval (a,∞). Thus for some $0 \leq k \leq m-1$ and some $C \neq 0$

$$G(x) \sim Cx^k \quad \text{as} \quad x \to \infty.$$

This implies that G has q.a.b. of order $k \leq m-1$ related to x^k, $x \to \infty$. The Structural Theorem ([2]) implies that S has q.a.b. of order $k-m$ related to x^{k-m}, $x \to \infty$.

Let us remember that the distribution $\delta_a^{(j)}$, $a > 0$, $j \in \mathbb{N}_0$, has a q.a.b. of order $-(j+1)$ related to $\frac{1}{x^{j+1}}$, $x \to \infty$. We use this fact in the considerations which follow.

Let g be a l.i.f. on $\mathbb{R} \setminus \{0\}$ equal to zero outside of some interval $[0,a]$, $a > 0$, such that

$$g(x) \sim x^{\alpha} L(x) \quad \text{as} \quad x \to 0^+,$$

where $\alpha \in \mathbb{R}$ and L is a s.v.f.

This function can be identified with the distribution S defined by (see [3], p.13):

$$\langle S,\phi\rangle := \int_0^a g(x)\phi(x)\,dx, \quad \phi \in \mathcal{S}, \quad \text{if} \quad \alpha > -1 \tag{3.1}$$

and

$$\langle S,\phi\rangle := \int_0^a g(x)\left(\phi(x)-\phi(0)-\ldots-\frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right)dx \tag{3.2}$$

if $(n+1) < \alpha \leq -n$.

PROPOSITION 5. *The distribution S defined by (3.1) ($\alpha > -1$) has q.a.b. of order -1 related to $\frac{1}{x}$, $x \to \infty$, and the one defined by (3.2) ($\alpha \leq -1$) has q.a.b. of order $-(n+1)$ related to $\frac{1}{x^{n+1}}$, $x \to \infty$, where n is chosen so that $-(n+1) < \alpha \leq -n$.*

P r o o f. The case $\alpha > -1$ is obvious.

Let $-(n+1) < \alpha \leq -n$, $n \in \mathbb{N}$. Then for $\phi \in \mathcal{S}$ we have

$$\langle k^{n+1}S(kx),\phi(x)\rangle = k^n \int_0^a g(x)\left(\phi\left(\frac{x}{k}\right) - \phi(0)-\ldots-\right.$$

$$\left. - \frac{x^{n-1}}{k^{n-1}(n-1)!}\phi^{(n-1)}(0)\right)dx = k^n \int_0^a g(x)\,\frac{1}{n!}\left(\frac{x}{k}\right)^n \phi^{(n)}\left(\frac{\xi}{k}\right)dx$$

$0 < \xi < ak$, hence

$$\lim_{k\to\infty} \langle k^{n+1}S(kx),\phi(x)\rangle = \frac{(-1)^n}{n!}\cdot\langle \delta^{(n)},\phi\rangle \cdot \int_0^a x^n g(x)\,dx \quad \text{as} \quad x \to \infty.$$

IV. Let us suppose additionally that L is also slowly varying at zero (see [5], p.11).. As before, α is a real and n natural number.

We analyze the distribution $R(x) = (x^{\alpha}L(x))_{+}$ defined in the following way:

$$(4.1) \qquad \langle R,\phi\rangle = \int_0^\infty x^{\alpha}L(x)\phi(x)\,dx \quad \text{if} \quad \alpha > -1\ ;$$

$$(4.2) \qquad \langle R,\phi\rangle = \int_0^\infty x^{\alpha}L(x)\left(\phi(x)-\phi(0)-\ldots 0\ \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right)dx$$

if $-(n+1) < \alpha < -n$ and

$$(4.3) \qquad \langle R,\phi\rangle = \int_0^\infty x^{\alpha}L(x)\left(\phi(x)-\phi(0)-\ldots-\frac{x^{n-2}}{(n-2)!}\phi^{(n-2)}(0) - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)H(a-x)\right)dx \quad \text{if} \quad \alpha = -n.$$

(The letter ϕ denotes always a test function from $\mathcal{S}$.)

PROPOSITION 6. *The distribution* $R(x) = (x^{\alpha}L(x))_{+}$ *defined above has q.a.b. of order* α *related to* $\rho(x) = x^{\alpha}L(x)$, $x\to\infty$, *if* $\alpha \notin Z_{-}$ *and related to* $\rho_1(x) = x^{\alpha}L^{*}(x)$ *if* $\alpha \in Z_{-} = \{-1,-2,\ldots,\}$.

P r o o f. The statement is obvious for $\alpha > -1$. Let now $-(n+1) < \alpha < -n$. Then by [1] Theorem 6, we have

$$\left\langle \frac{R(kx)}{k^{\alpha}L(k)},\phi(x)\right\rangle = \frac{1}{k^{\alpha}L(k)}\int_0^\infty (kx)^{\alpha}L(kx)\left(\phi(x) - \ldots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right)dx \sim \frac{L(k)}{L(k)}\int_0^\infty x^{\alpha}\left(\phi(x)-\phi(0) - \ldots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right)dx \quad \text{as} \quad x\to\infty .$$

Hence

$$\lim_{k\to\infty}\left\langle \frac{R(kx)}{k^{\alpha}L(k)},\phi(x)\right\rangle = \langle x_{+}^{\alpha},\phi\rangle = \Gamma(\alpha+1)\langle D^{n}f_{\alpha+n+1}\phi\rangle)$$

where x_{+}^{α} is defined by (4.2) for $L(x) = 1$, and this proves the Proposition for $\alpha \notin Z_{-}$. At last, the distribution $R_{-n}(x)$ defined by (4.3) has q.a.b. of order $\alpha = -n$ related to

$\frac{L^*(x)}{x^n}$, $x \to \infty$ (L^* is defined by ($\overline{2}.3$) , since:

$$\lim_{k\to\infty} \frac{1}{k^{-n}L^*(k)} < R_{-n}(kx), \phi(x) > =$$

$$= \lim_{k\to\infty} \{ \frac{1}{k^{-n+1}L^*(k)} (\int_0^a \frac{L(x)}{x^n} (\phi(\frac{x}{k}) - \phi(0) - \ldots - \phi^{(n-1)}(0)(\frac{x}{k})^{n-1}\frac{1}{(n-1)!}) dx$$

$$+ \int_a^\infty \frac{L(x)}{x^n} (\phi(\frac{x}{k}) - \phi(0) - \ldots - \phi^{(n-2)}(0)\ (\frac{x}{k})^{n-2} \frac{1}{(n-2)!}) dx) \} =$$

$$= \lim_{k\to\infty} \{ \frac{1}{L^*(k)} \int_0^{a/k} \frac{L(kx)}{x^n} (\phi(x) - \phi(0) - \ldots - \phi^{(n-1)}(0)\ \frac{x^{n-1}}{(n-1)!})\ dx) +$$

$$+ \int_{a/k}^\infty \frac{L(kx)}{x^n} (\phi(x) - \phi(0) - \ldots - \phi^{(n-2)}(0)\ \frac{x^{n-2}}{(n-2)!}) dx \} =$$

$$= \lim_{k\to\infty} \frac{1}{L^*(k)} \{ \int_0^a \frac{L(kx)}{x^n} (\phi(x) - \phi(0) - \ldots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0)) dx +$$

$$+ \int_{a/k}^a \frac{L(kx)}{x^n} \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) dx + \int_a^\infty \frac{L(kx)}{x^n} (\phi(x) - \phi(0) - \ldots$$

$$- \frac{\phi^{(n-2)}(0)}{(n-2)!} x^{n-2} dx \} = \lim_{k\to\infty} \frac{L(k)}{L^*(k)} \int_0^\infty \frac{1}{x^n} (\phi(x) - \phi(0) - \ldots - \frac{x^{n-2}}{(n-1)!} \phi^{(n-2)}(0) -$$

$$- H(a-x) \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0)) dx +$$

$$+ \lim_{k\to\infty} \frac{\phi^{(n-1)}(0)}{L^*(k)} \int_a^{ak} \frac{L(x)}{x} dx = 0 + < \delta^{(n-1)}, (-1)^{n-1}\phi > .$$

We have used repeatedly Theorem 6 from [1], relation $\lim_{k\to\infty} \frac{L(k)}{L^*(k)} = 0$ from [5], p.86 (Parameswaran [4]) and the fact that $L^*(k)$ is slowly varying, provided that $L(k)$ is.

V. In this section f denotes a measurable function on $\mathbb{R}$ with support in $[0,\infty)$ satisfying the a.b.

(5.1) $$f(x) \sim x^{\alpha_1} L_1(x) \quad \text{as} \quad x \to 0+$$

and L_1 is a s.v.f. at zero.

There is a question about the q.a.b. of the appropriate regularization of f, if f satisfies some additional conditions at infinity. Let $-n-1 < \alpha_1 \leq -n$. We denote by $\tilde{f}$ the following distribution

$$(5.2)\qquad \langle \tilde{f},\phi\rangle = \int_0^\infty f(x)\,(\phi(x)-\phi(0)-\ldots-\frac{x^{n-1}}{(n-1)!}\,\phi^{(n-1)}(0))\,dx$$

if $-n-1 < \alpha_1 < -n$ and

$$(5.3)\qquad \langle \tilde{f},\phi\rangle = \int_0^\infty f(x)\,(\phi(x)-\phi(0)-\ldots-\frac{x^{n-2}}{(n-2)!}\,\phi^{(n-2)}(0) - \frac{x^{n-1}}{(n-1)!}\,\phi^{(n-1)}(0)H(a-x))\,dx$$

for some $a > 0$ if $\alpha_1 = -n$.

We suppose that f satisfies the following condition

$$(5.4)\qquad x^n f(x) \quad \text{is integrable on } (a,\infty)\ .$$

Let us prove

PROPOSITION 7. *The distribution $\tilde{f}$ from (5.2), respectively (5.3), has q.a.b. of order $-n-1$ related to $\frac{1}{x^{n+1}}$, respectively, of order $-n-1$ related to $\frac{1}{x^{n+1}}$, provided that both (5.1) and (5.4) hold.*

P r o o f. We start from the equality $(-n-1 < \alpha_1 < -n)$

$$k^{n+1} \langle f(kx),\phi(x)\rangle =$$

$$= k^n \int_0^\infty f(x)\,(\,\phi(\tfrac{x}{k}) - \ldots - \frac{x^{n-1}}{(n-1)!}\,\frac{\phi^{(n-1)}(0)}{k^{n-1}})\,dx = k^n\,(\int_0^a + \int_a^\infty\,)\,.$$

Now by Proposition 5 the first integral in the last bracket behaves as $\frac{c}{k^{n+1}}$ as $k\to\infty$ for some $c \neq 0$ and the second by the Lebesgue theorem behaves as

$$\frac{\phi^{(n)}(0)}{k^n} \int_a^\infty x^n f(x)\,dx \ .$$

This finishes the proof of the first part of Proposition 7.

The case $\alpha_1 = -n$ is similar, so we omit the proof.

As a direct consequence of Proposition 7 we get

PROPOSITION 8. *Let* f *satisfy (5.1) for* $-n-1 < \alpha_1 < -n$ *and*

(5.5) $$f(x) \sim x^{\alpha_2} L_2(x) \quad \text{as} \quad x \to \infty \quad (\alpha_2 < -n)$$

where L_2 *is a s.v.f. at infinity.*

Then $\tilde{f}$ *defined by (5.2)* $(-n-1 < \alpha_1 < -n)$*, respectively by (5.3)*$(\alpha_1 = -n)$ *has q.a.b. of order* $-n-1$ *related to* $\frac{1}{x^{n+1}}$*, respectively, of order* $-n$ *related to* $\frac{1}{x^n}$.

Observe that this q.a.b. does not depend on the functions L_1 and L_2. At the end we give

PROPOSITION 9. *Let* f *satisfy (5.1) for* $-n-1 < \alpha_1 < -n$ *and*

(5.6) $$f(x) \sim x^{\nu} L_2(x) \quad \text{as} \quad x \to \infty$$

for $\nu > 0$ *where* L_2 *is s.v.f. at infinity.*

Then $\tilde{f}$ *defined by (5.2) has q.a.b. of order* ν *related to* $x^{\nu} L_2(x)$.

P r o o f. We have

(5.7) $$\frac{1}{k^{\nu} L_2(k)} < \tilde{f}(kx), \phi(x) > = \frac{1}{k^{\nu+1} L_2(k)} < \tilde{f}(x), \phi(\frac{x}{k}) > =$$

$$= \frac{1}{k^{\nu+1} L_2(k)} \int_0^a f(x)\left(\phi(\frac{x}{k}) - \phi(0) - \ldots - (\frac{x}{k})^{n-1} \frac{\phi^{(n-1)}(0)}{(n-1)!}\right) dx +$$

$$+ \frac{1}{k^{\nu} L_2(k)} \int_a^\infty f(x)\, \phi(\frac{x}{k})\, dx \ .$$

The first part on the right side of (5.7) tends to zero as $k \to \infty$. So we have to prove that

$$\frac{1}{k^{\nu}L_2(k)} \int_{a/k}^{\infty} f(kx)\phi(x)dx \to \int_0^{\infty} x^{\nu}L_2(x)\phi(x)dx \quad \text{as} \quad k \to \infty .$$

From the assumption (5.6) we have that for every $\varepsilon > 0$ there exists $M > 0$ such that

$$|f(x) - x^{\nu}L_2(x)| < \varepsilon x^{\nu}L_2(x) + M \quad \text{for} \quad x \geq 1 .$$

This implies

$$|f(kx) - (xk)^{\nu}L_2(kx)| < \varepsilon(xk)^{\nu}L_2(kx) + M \quad \text{for} \quad x \geq 1 .$$

Thus we have

$$\left| \frac{1}{k^{\nu}L_2(k)} \int_{a/k}^{\infty} f(kx)\phi(x)dx - \frac{1}{k^{\nu}L_2(k)} \int_0^{\infty} f(kx)^{\nu}L_2(kx)\phi(x)dx \right| <$$

$$< \frac{1}{k^{\nu}L_2(k)} \int_{a/k}^{\infty} |f(kx) - (kx)^{\nu}L_2(kx)|\,|\phi(x)|dx +$$

$$+ \frac{1}{k^{\nu}L_2(k)} \int_0^{a/k} (kx)^{\nu}L_2(kx)|\phi(x)|dx < \frac{\varepsilon}{L_2(k)} \int_0^{\infty} x^{\nu}L_2(kx)|\phi(x)|dx +$$

$$+ \frac{M}{k^{\nu}L_2(k)} \int_0^{\infty} |\phi(x)|dx + \frac{1}{L_2(k)} \int_0^{a/k} x^{\nu}L_2(kx)|\phi(x)|dx$$

From these inequalities the assertion follows.

Added in proof: Proposition 9 with $\nu > -1$ is proved by S.Pilipović in the paper:"On the quasiasymptotic behaviour of Stieltjes transform of distributions", Lemma 3(to appear).

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REZIME

KVAZIASIMPTOTSKO PONAŠANJE NEKIH DISTRIBUCIJA

Data je eksplicitno kvaziasimptotika u beskonačnosti nekih tipova temperiranih distribucija.