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S-ASYMPTOTIC OF TEMPERED AND K_1'-DISTRIBUTIONS, PART I

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ABSTRACT

It is proved that if $f \in K_1'(R)$ ($f \in S'(R)$) has the S-asymptotic in $D'(R)$ related to a function $c(h) \in \Sigma_e(R)$ ($c(h) \in \Sigma_p(R)$), $h \to \infty$, then f has the S-asymptotic in $K_1'(R)$ (in $S'(R)$) related to c(h). For the same assertion in the n-dimensional case n > 1, some additional assumptions are needed.

1. NOTATION AND INTRODUCTION

We denote by R and N the sets of real and natural numbers; $N_0 = N \cup \{0\}$. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in R^n$, then $\langle x,y \rangle = x_1y_1 + \ldots + x_ny_n$; $\|x\|^2 = x_1^2 + \ldots + x_n^2$; $|x| = |x_1| + \ldots + |x_n|$; $x \geq 0 \iff x_i \geq 0$, $i = 1,\ldots,n$; $x \to \infty \iff x_i \to \infty$, $i = 1,\ldots,n$; $R_+^n = \{x \in R^n, x \geq 0\}$.

For the definitions and properties of spaces $D(R^n)$, $D_K(R^n)$, $D_K^m(R^n)$, $E(R^n)$, $S(R^n)$, $D'(R^n)$ and $S'(R^n)$ we refer the re-

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ader to [4]. For the definition of spaces $K_1(R^n)$ and $K_1'(R^n)$ see [1] or [5].

The space of all the real valued functions c(h) different from zero for $h \in R^n_+$ is denoted by $\Sigma(R^n)$. We assume without losing generality, that c(h) are positive and equal to 1 in $R^n_+ \setminus (R^n_+ + a)$ where $a \in R^n_+$ depends on c(h) and $R^n_+ + a = \{x + a,\ x \in R^n_+\}$.

By $\Sigma_e(R^n)$ $(\Sigma_p(R^n))$ is denoted a subset of $\Sigma(R^n)$ such that $c(h) \in \Sigma_e(R^n)$ $(c(h) \in \Sigma_p(R^n))$ iff for some $C > 0$, $d > 0$, $k > 0$ and $h_0 = (h_{01}, \ldots, h_{0n}) \in R^n_+$:

(1) $$c(h+r) \leq C\, c(h)\exp(k|r|) \quad (c(h+r) \leq C\, c(h)(1+|r|)^k),$$

$$h > h_0,\ r \in R^n;$$

(2) $$c(h)\exp(k|h|) \geq d, \quad (c(h)(1+|h|)^k \geq d),\ h > h_0.$$

Obviously, $\Sigma_p(R^n) \subset \Sigma_e(R^n)$. A detailed analysis of conditions (1) and (2) is given in the paper following this one ([2]).

We defined in [3] the S-asymptotic of distributions in a cone Γ. For the sake of simplicity we restrict ourself to the cone $\Gamma = R^n_+$. Let us recall from [3] that a distribution $T \in D'(R^n)$ has the S-asymptotic in the cone R^n_+ related to some $c(h) \in \Sigma(R^n)$, if there exists the limit

(*) $$\lim_{h\to\infty} T(x+h)/c(h) = S(x) \text{ in } D'(R^n).$$

In this case we write $T(x+h) \overset{s}{\sim} c(h)S(t)$.

We give in [3] a full characterization of the limit distributions S. Namely, it was proved in [3] that S must be of the form

$$S(x) = C\exp(\langle r,x\rangle) \text{ for some } C \in R \text{ and some } r \in R^n.$$

Thus, $S \in K_1'(R^n)$, and there arises a natural question: If $T \in K_1'(R^n)$

can the limit (*) be extended from $D(R^n)$ onto $K_1(R^n)$? If $T \in S'(R^n)$ and $r = 0$, there is a similar question for $S'(R^n)$, instead of $K_1'(R^n)$. These questions are important for applications since spaces $K_1'(R^n)$ and especially $S'(R^n)$ have many useful properties.

We prove in this article that if $T \in K_1'(R)$ ($T \in S'(R)$) has the S-asymptotic in the cone R_+ related to some $c(h)) \in \Sigma_e(R)$, ($c(h) \in \Sigma_p(R)$) then the limit (*) exists for every $\phi \in K_1(R)$ ($\phi \in S(R)$). If $n > 1$, we have to assume some additional conditions which imply the same assertion for the multidimensional case.

2. Using the representation theorem for elements from $K_1'(R^n)$ (see [4]):

(3) $$T \in K_1'(R^n) \Longleftrightarrow (\exists m \in N_0)(\exists k > 0)(\exists F_i \in L_\infty(R) \cap C(R), \; |i| \leq m)$$

$$(f = \sum_{|i| \leq m} (F_i(x)\exp(k|x|))^{(i)}),$$

we obtain directly

PROPOSITION 1. *If* $T \in K_1'(R^n)$, *then for some* $k > 0$ $\{T(x+h)\exp(-k|h|),\ h \in R^n\}$ *is bounded in* K_1'.

We have noticed that a corresponding assertion for $S'(R^n)$ is given in [4, Chapitre VII, Théorème VI, 4°].

A complex valued function $r(x)$ defined on R^n is called a rapidly exponentially decreasing function if for every $k > 0$

$$r(x)\exp(k|x|) \to 0 \text{ as } |x| \to \infty.$$

Similarly as in [4, Chapitre VII, Théorème VI, 4°] for $S'(R^n)$, we prove the next

THEOREM 2. *Let* $T \in \mathcal{D}'(\mathbb{R}^n)$. *If for every rapidly exponentially decreasing function* $r(x)$ *the set* $\{r(h)T(x+h), h \in \mathbb{R}^n\}$ *is bounded in* $\mathcal{D}'(\mathbb{R}^n)$, *then* $T \in \mathcal{K}'(\mathbb{R}^n)$.

PROOF. Let K be an arbitrary compact set in $\mathbb{R}^n$. For every $\phi \in \mathcal{D}_K(\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}^n)$

$$h \to r(h) \langle T(x+h), \phi(x) \rangle, \quad h \in \mathbb{R}^n,$$

is a bounded function. This implies that for some $k_1 = k_1(\phi) > 0$ and some $C = C(\phi) > 0$

$$|\langle T(x+h), \phi(x) \rangle| \leq C \exp(k_1 |h|), \; h \in \mathbb{R}^n. \tag{4}$$

If $h \in \mathbb{R}^n$ is fixed, then by

$$\log^+ |\langle T(x+h), \phi(x) \rangle| \,/\, (1+|h|), \phi \in \mathcal{D}_K(\mathbb{R}^n),$$

a continuous linear functional on $\mathcal{D}_K(\mathbb{R}^n)$ is defined. It follows from (4) that $\{\phi \to \log^+ |\langle T(x+h), \phi(x) \rangle| \,/\, (1+|h|), h \in \mathbb{R}^n\}$ is a bounded family of continuous functions on $\mathcal{D}_K(\mathbb{R}^n)$. By the same arguments as in [4, Tome II, p. 97], we conclude that for some $k > 0$, which does not depend on $\phi \in \mathcal{D}_K(\mathbb{R}^n)$, the set $\{\langle T(x+h)\exp(-k|h|), \phi(x) \rangle, h \in \mathbb{R}^n\}$, is bounded for every $\phi \in \mathcal{D}_K(\mathbb{R}^n)$, i.e. for every $\phi \in \mathcal{D}(\mathbb{R}^n)$. This implies (see [4, Chapitre VI, Théorème XXII]) that for a given open bounded set $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, there exist a compact neighbourhood of zero K and $m \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{D}_K^m(\mathbb{R}^n)$, $\{x \to (T(t+h) * \phi(t))(x)/\exp(k|h|), h \in \mathbb{R}^n\}$ is a bounded family of continuous bounded functions on Ω.

Since $(T(t+h) * \phi(t))(x) = (T * \phi)(x+h)$, on setting $x = 0$, we obtain that

$$h \to (T * \psi)(h)/\exp(k|h|), \; h \in \mathbb{R}^n,$$

is a bounded function on $\mathbb{R}^n$ for any $\psi \in \mathcal{D}_K^m(\mathbb{R}^n)$.

Now, by [4, VI, 6; 22] we obtain

$$(5) \qquad T = \Delta^N(\gamma E * T) - \psi * T$$

where E is the fundamental solution of $\Delta^N E = \delta$ (Δ is the Laplacian) $\gamma \in D_K$, $\gamma \equiv 1$ in a neighbourhood of 0 and $\psi \in D_K(R^n)$. If N is sufficiently large $\gamma E \in D_K^m(R^n)$. Thus, T is of the form (3). This completes the proof.

Let us notice that if $T \in D'(R^n)$ and $\{T(x+h)\exp(-k|h|), h \in R^n\}$ is bounded in $D'(R^n)$ for some $k > 0$, then for every rapidly exponentially decreasing function $r(x)$, $\{r(h)T(x+h), h \in R^n\}$ is bounded in $D'(R^n)$, as well.

The following theorem will be proved first in the one--dimensional case, because in the n-dimensional case more assumpions are needed.

THEOREM 3. *Let* $T \in K_1'(R)$ ($T \in S'(R)$) *and* $c(h) \in \Sigma_e(R)$ ($c(h) \in \Sigma_p(R)$). *If* $\{T(x+h)/c(h),\ h > a\}$ *is bounded in* $D'(R)$, *then this family is bounded in* $K_1'(R)$ (*in* $S'(R)$).

PROOF. We prove the theorem for $T \in K_1'(R)$, because for $T \in S'(R)$ it can be done in a similar way. (We have to replace $\exp(k|\cdot|)$ by $(1+|\cdot|)^k$.)

Using the last part of the proof of Theorem 2, we obtain that for some $m_1 \in N_0$ and some compact neighbourhood of zero, K_1,

$$h \to (T * \psi)(h)/c(h), \quad h > 0$$

is a bounded function for every $\psi \in D_{K_1}^{m_1}(R)$. Since $T \in K_1'(R)$ it holds that for some $k > 0$, $m_2 \in N_0$ and some compact neighbourhood of zero K_2,

$$h \to (T * \psi)(h)/\exp(k|h|), \ h \in R ,$$

is a bounded function for every $\psi \in D^{m_2}_{K_2}(R)$. Thus, by taking in (5) N sufficiently large ($\Delta = d^2/dx^2$) and $K = K_1 \cap K_2$, we obtain that for some $m \in N_0$

$$(6) \qquad T = \sum_{i=0}^{m} F_i^{(i)},$$

where F_i, $i = 0,\dots,m$, are continuous functions on R such that for some $M_1 > 0$, $M_2 > 0$ and $k > 0$

$$(7) \qquad \sup\{|F_i(x)/c(x)|,\ \ x > 0,\ \ i = 0,\dots, m\} \le M_1,$$

$$(8) \qquad \sup\{|F_i(x)|/\exp(k|x|),\ x \in R,\ i = 0,\dots, m\} \le M_2.$$

Let $\phi \in K_1(R)$ and $h > h_0$ (see (1)) be fixed. We put

$$I_i(h,\phi) = I_i = \int_{-\infty}^{\infty} (|\phi^{(i)}(x)||F_i(x+h)|/c(h))dx =$$

$$= (\int_{-\infty}^{-h} + \int_{-h}^{\infty})(|\phi^{(i)}(x)||F_i(x+h)|/c(h))dx =$$

$$= I_i(-\infty,-h) + I_i(-h,\infty),\ i = 0,\dots,m,\ h > h_0.$$

If $x \in (-\infty,-h)$, $|x+h| = |x| - h$ and by (8) and (2), we obtain

$$I_i(-\infty,-h) \le d^{-1} \int_{-\infty}^{-h} |\phi^{(i)}(x)|\exp(2k|x|)$$

$$\cdot\ |F_i(x+h)/\exp(k|x+h|)|\,dx \le$$

$$\le M\ d^{-1} \int_{-\infty}^{\infty} |\phi^{(i)}(x)\ \exp(k|x|)dx.$$

From the definition of space $K_1(R)$, it follows that the last integral is finite.

Because of (1) and (7), we obtain that for some $k_1 > 0$,

$$I_i(-h,\infty) \le \int_{-h}^{\infty} |\phi^{(i)}(x)||F_i(x+h)/c(x+h)|\exp(k_1|x|)dx$$

$$\le M_1 \int_{-\infty}^{\infty} |\phi^{(i)}(x)|\exp(k_1|x|)dx < \infty.$$

Since

$$|< T(x+h)/c(h),\phi(x) >| \le \sum_{i=0}^{m} I_i(h,\phi),$$

from the preceding inequalities we obtain that for some $A > 0$ which does not depend on $h > h_0$

$$|< T(x+h)/c(h),\phi(x) >| \le A.$$

Thus the proof is complete.

THEOREM 4. *Let* $T \in D'(R)$ *such that* $\operatorname{supp} T \subset [0,\infty)$, *and let* $c(h)$ *satisfy condition (2) and*

(1)* *There exist* $C > 0$, $k > 0$, $h_0 > 0$ *such that*

$$c(h+r) \le C\, c(h)\exp(k\, r)\ \ (c(h+r) \le C\, c(h)(1+ r)^k),$$

$$h \ge h_0,\ r > 0.$$

Moreover, let $\{T(x+h)/c(h);\ h > a\}$ *be bounded in* $D'(R)$. *Then, this family is bounded in* $K_1'(R)$ *(in* $S'(R)$*).*

PROOF. The proof is similar to the proof of Theorem 3. We have only to notice that in (6) functions f_i, $i=0,\dots,m$, have their supports in $[A,\infty)$ for some $A \in R$. Thus, instead of (7) and (8) we have to use

(7)* $\sup\{|F_i(x)/c(x)| : x > A,\ i = 0,\dots,m\} < M_1,$

and the fact

$$I_i(h,\phi) = I_i = \int_{A-h}^{\infty} |\phi^{(i)}(x)||F_i(x+h)/c(h)|dx,$$

$$i = 0,\ldots, m$$

(see the proof of Theorem 3).

In order to extend the preceding theorem for a multi-dimensional case, we have to introduce the following notation.

We denote by Λ the set of all n-th class variations of elements $\{-1,1\}$. If $(a_1,\ldots,a_n) \in \Lambda$, then we put

$$\Gamma(a_1,\ldots, a_n) = \{h \in R^n, \sum_{i=1}^{n} \operatorname{sgn}(a_i \cdot h_i) = n\}.$$

(This means if $a_i = 1$, ($a_i = -1$), then $h_i > 0$ ($h_i < 0$).) For example

$$\Gamma(1,\ldots,1) = R^n_+ \quad \text{and} \quad \Gamma(-1,\ldots,-1) = R^n_- .$$

Let

(9) $c(h) = c_1(h_1) \ldots c_n(h_n)$ where

$$c_i(h_i) \in \Sigma_e(R)(c_i(h_i) \in \Sigma_p(R)),\ i = 1,\ldots,n.$$

Obviously, $c(h) \in \Sigma_e(R^n)$ ($c(h) \in \Sigma_p(R^n)$).

Let $(a_1,\ldots, a_n) \in \Lambda$ be given. We denote by j_i, $i = 1,\ldots, r$ components of $(a_1, \ldots, a_n)$ equal to 1, and those which are equal to -1 by s_i, $i = 1,\ldots, m$ $(r+m = n)$.

If $h \in \Gamma(a_1,\ldots, a_n)$ and $k > 0$, we put

$$c^k_{(a_1,\ldots,a_n)}(h) = c_{j_1}(h_{j_1}) \ldots c_{j_r}(h_{j_r}) \cdot$$

$$\cdot \exp(k(|h_{s_1}| + \ldots + |h_{s_m}|))$$

$$(c^k_{(a_1,\dots,a_n)}(h) = c_{j_1}(h_{j_1})\dots c_{j_r}(h_{j_r})((1+|h_{s_1}|)\dots(1+|h_{s_m}|))^k).$$

THEOREM 5. *Let* $T \in K_1'(R^n)$ $(T \in S'(R^n))$ *and* $c(h)$ *be of the form (9). If there exists* $k>0$ *such that for every* $(a_1,\dots,a_n) \in \Lambda$

$$\{T(x+h)/c^k_{(a_1,\dots,a_n)}(h),\ h \in \Gamma(a_1,\dots,a_n)\}$$

is bounded in $D'(R^n)$, *then* $\{T(x+h)/c(h),\ h > 0\}$ *is bounded in* $K_1'(R^n)$ $(S'(R^n))$.

We notice that from Proposition 1 it follows that for every $T \in K_1'(R^n)$ there exists $k > 0$ such that $\{T(x+h)/\exp(k|h|),\ h \in R^n\}$ is bounded in $K_1'(R^n)$. A similar assertion holds for $S'(R^n)$. Thus, for $n = 1$ Theorem 5 reduces into Theorem 3. (If $h \in R^n_+$ then $c^k_{(1,\dots,1)}(h) = c(h)$.)

PROOF OF THEOREM 5. If we repeat the arguments of the first part of Theorem 3, 2^n-times, we obtain that for some $m \in N_0$

$$T = \sum_{|i|\le m} (F_i)^{(i)},$$

where F_i, $|i|\le m$ are continuous functions such that for every $(a_1,\dots,a_n) \in \Lambda$

$$\sup\{|F_i(x)/c^k_{(a_1,\dots,a_n)}(x)|,\ x \in \Gamma(a_1,\dots,a_n),\ |i| \le m\} < M_{(a_1,\dots,a_n)}. \tag{10}$$

We put, for the given $\phi \in K_1(R^n)$ and $h \in R^n_+$:

$$I_i(h,\phi) = \int_{R^n} (|\phi^{(i)}(x)||F_i(x+h)|/c(h))dx =$$

$$= \left\{ \sum_{(a_1,\ldots,a_n)\in\Lambda} \int_{x+h\in\Gamma(a_1,\ldots,a_n)} \right\} (|\phi^{(i)}(x)|\cdot$$

$$\cdot\ |F_i(x+h)|/c(x))dx,\ \ |i|\le m.$$

As in Theorem 3, one can show that every member of the last series is bounded by some constant which does not depend on h. Since

$$|< T(x+h)/c(h),\phi(x) >| \le \sum_{|i|\le m} I_i(h,\phi),$$

the theorem is proved.

Now it is easy to prove the following three theorems.

THEOREM 6. *Let* $T \in K_1'(R)$ $(T \in S'(R))$ *and* $c(h) \in \Sigma_e(R)$ $(c(h) \in \Sigma_p(R))$. *If there exists the limit*

$$\lim_{h\to\infty} < T(x+h)/c(h),\phi(x) > = < S(x),\phi(x) >,\ \phi \in D(R),$$

then this limit exists for every $\phi \in K_1(R)$ *(for every* $\phi \in S(R)$. *In particular* , $S \in K_1'(R)$ $(S \in S'(R))$.

PROOF. Since $D(R)$ is dense in $K_1(R)$ (in $S(R)$) we hawe to use Theorem 3 and the Banach-Steinhaus Theorem.

THEOREM 7. *Let* T *and* c(h) *satisfy the conditions of Theorem 4. If there exists the limit*

$$\lim_{n\to\infty} < T(x+h)/c(h),\phi(x) > = < S(x),\phi(x) > ,\phi \in D(R),$$

then this limit exists for every $\phi \in K_1(R)$ *(for every* $\phi \in S(R)$). *In particular* , $S \in K'(R)$ $(S \in S'(R))$.

In the same way, using Theorem 5, one can prove the following

THEOREM 8. *Let* $T \in K_1'(R^n)$ $(T \in S'(R^n))$, $c(h)$ *be of the form (9) and*

$$\lim_{h\to\infty} \langle T(x+h)/c(h),\phi(x)\rangle = \langle S(x),\phi(x)\rangle,\ \phi \in D(R).$$

If for some $k > 0$ *and every* $(a_1,\dots,a_n) \in \Lambda \setminus (1,\dots,1),(-1,\dots,-1)$, *the sets*

$$\{T(x+h)/c^k_{(a_1,\dots,a_n)}(h);\ h \in \Gamma(a_1,\dots,a_n)\} ,$$

are bounded in $D'(R^n)$, *then*

$$T(x+h)/c(h) \textit{ converges to } S(x) \textit{ in } K_1'(R^n)$$

(*in* $S'(R^n)$) *as* $h \to \infty$.

Particularly, $S \in K_1'(R^n)$ $(S \in S'(R^n))$.

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REZIME

S-ASIMPTOTIKA TEMPERIRANIH I K_1'-DISTRIBUCIJA.
I DEO

Dokazano je da ako $T \in K_1'(R)$ $(S'(R))$ ima S-asimptotiku u $D'(R)$ u odnosu na $c(h) \in \Sigma_e(R)$ $(c(h) \in \Sigma_p(R))$, $h \to \infty$, tada f ima S-asimptotiku u $K_1'(R)$ (u $S'(R)$) u odnosu na $c(h)$. U slučaju da je $n > 1$ za slično tvrdjenje u konusu R_+^n dodatni uslovi su uvedeni.