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S-ASYMPTOTIC OF TEMPERED AND K_1'-DISTRIBUTIONS. PART II. FUNCTIONS FOR COMPARISON

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ABSTRACT

Sets of functions $\Sigma_e(R^n)$ and $\Sigma_p(R^n)$ were introduced in [2], where we studied the S-asymptotic of the $K_1'(R^n)$ and $S'(R^n)$ - distributions. In this paper we investigate these two sets of functions by using the theory of slowly varying functions (n=1).

INTRODUCTION

We use the notation given in [2]. Let us recal that $\Sigma(R)$ is the set of all real valued positive functions which are equal to 1 in $(-\infty,a)$ for some $a > 0$ (which depends on c(h)). $\Sigma_p(R)$ is a set of all c(h) from $\Sigma(R)$ for which there exist $C > 0$, $k > 0$, $h_0 > 0$ and $d > 0$ such that

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(P.1) $c(h+r) \leq C\, c(h)(1+|r|)^k$, $h > h_0$, $r \in R$;

(P.2) $c(h)(1+ h)^k \geq d$, $h > h_0$.

Similarly, $\Sigma_e(R)$ is a subset of $\Sigma(R)$ such that $c(h) \in \Sigma_e(R)$, if there exists $C > 0$, $k > 0$, $h_0 > 0$ and $d > 0$ such that

(E.1) $c(h+r) \leq C\, c(h) \exp(k|r|)$, $h > h_0$, $r \in R$;

(E.2) $c(h) \exp(k|h|) \geq d$, $h > h_0$,

The slowly varying function L is a positive continuous function defined in some neighbourhood of ∞ for which

(1) $\lim_{t\to\infty} L(\lambda t)/L(t) = 1$, $\lambda > 0$ ([1]).

It is enough to assume instead of continuity that L is measurable but because of [1,1.4.] it is not the restriction.

We need the following properties of a slowly varying function:

(2) For every $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that

$$h^\varepsilon > L(h) > h^{-\varepsilon} \quad \text{for } h > h_\varepsilon \quad [1];$$

(3) If $\lambda \in [a,b]$, $0 < a < b < \infty$, then the limit (1) is uniform in λ([1]);

(4) For every $\varepsilon > 0$ there exists $h_\varepsilon > 0$ such that

$$L(\lambda h) \leq L(h)\lambda^\varepsilon, \; h > h_0, \; \lambda > 1+\varepsilon \quad ([4]).$$

Let $c(h) \in \Sigma(R)$, $T \in D'(R)$ and $S \in D'(R)$. If

(5) $\lim_{h\to\infty} < T(x+h)/c(h), \phi(x) > \; = \; < S(x), \phi(x) >$, $\phi \in D(R)$

then we say that T has the S-aymptotic related to c(h) with the limit S as $h \to \infty$ ([3]). We proved in [2] that if $T \in K_1'(R)$ ($T \in S'(R)$) and $c(h) \in \Sigma_e(R)$ ($c(h) \in \Sigma_p(R)$) then the limit (5) may be extended from D(R) onto $K_1(R)$ (onto S(R)).

Since these assertions are important in an investigation of the notion of the S-asymptotic, and conditions (P.1), (P.2), ((E.1), (E.2), are also theoretical, in this paper we investigate $\Sigma_p(R)$ and $\Sigma_e(R)$. We give explicit representations for the elements of these spaces if they satisfy resonable assumptions.

2. SET $\Sigma_p(R_+)$

Let us suppose that $c(h) \in \Sigma(R)$ and that this function is a continuous one in some neighbourhood of ∞. If the limit (5) exists with $S \neq 0$, then, it was proved in [2], c(h) must be of the form

$$(6) \quad c(h) = L(\exp(h)) \exp(\alpha h), \quad h > \bar{h}$$

where $\alpha \in \mathbf{R}$, $\bar{h} > 0$, are suitable constants and L is a suitable slowly varying function. Moreover, if we also assume that $T \in S'(R)$ (in (5)), then, by the structural theorem for elements from $S'(R)$, it follows that $\alpha \leq 0$ in (6). Namely, $\alpha > 0$ in (6) would imply that $S = 0$ in (5).

Moreover, if we suppose that $S \in S'(R)$ then in (6) α must be equal to 0. This follows from [3, Proposition 3]. Thus, we have proved

PROPOSITION 1. *Suppose that* c(h) *is a function from* $\Sigma(R)$ *which is continuous in some neighbourhood of* ∞. *Moreover, suppose that* $T \in S'(R)$ *and that (5) holds with* $S \neq 0$, $S \in S'(R)$.

Then $\alpha = 0$ *in* (6) *and* (P.2) *holds if for some* $h_1 > 0$, $C > 0$ *and* $\beta \in \mathbf{R}$ $L(\exp(h)) > Ch^{\beta}$ *if* $h > h_1$.

PROPOSITION 2. *Let* $c(h) \in \Sigma(R)$ *be of the form*

$$(7) \qquad c(h) = h^{\nu}L(h),\ h > h_1, \quad c(h) = 1,\ h \leq h_1,$$

where $\nu \in R$, $h_1 > 0$ *and* L *is a slowly varying and monotonous function for* $h > h_1$. *Then* c(h) *satisfies* (P.1).

PROOF. Supoose that L is non-decreasing.

Let $r+h > h_1$ and $h > h_1$. Since $r > h_1 - h$, for $h_1 - h < r < 0$ we have

(8) $$(h+r)^\nu L(h+r) \leq (h+r)^\nu L(h).$$

Let $0 \leq r \leq h$. Since $L(h+r) = L(h(1 + r/h))$ and $1 \leq (1 + r/h) \leq 2$ we obtain that for some $h_2 > h_1$ and some $C_1 > 0$

(9) $$(h+r)^\nu L(h+r) \leq C_1 (h+r)^\nu L(h),\ h > h_2,$$

holds.

Let $r > h$. Then $(1 + r/h) > 2$ and by (4), we obtain that for some ε, $0 < \varepsilon < 1$, and some $h_\varepsilon > h_1$

(10) $$(h+r)^\nu L(h+r) = (h+r)^\nu L(h(1 + r/h)) \leq$$
$$\leq (h+r)^\nu L(h)(1 + r/h)^\varepsilon \leq (h+r)^\nu L(h)(1 + |r|/h_1)^\varepsilon$$
$$\leq (h + r)^\nu L(h)(1 + |r|)^\varepsilon.$$

Now, if we prove that for some $C_2 > 0$, $k_1 > 0$ and $h_3 > h_1$

(11) $$(h+r)^\nu \leq C_2 h^\nu (1+r)^k,\ h > h_3,\ r > h_1 - h$$

holds,then (8), (9) and (10) imply that for some $h_0 > 0$, $k > 0$ and $C > 0$

(12) $$(h+r)^\nu L(h+r) \leq C h^\nu L(h)(1 + |r|)^k,\ h > h_0,\ r > h_1 - h.$$

If $\nu \geq 0$ then (11) holds trivially. If $\nu > 0$, then (11) follows from the inequality

$$h^{-\nu}/(h+r)^{-\nu} \leq (h+r+|r|)^{-\nu}/(h+r)^{-\nu} \leq$$

$$\leq 2^{-\nu}((h+r)^{-\nu}+ |r|^{-\nu})/(h+r)^{-\nu} \leq$$

$$\leq C_2(1 + |r|)^{-\nu}, \ h > h_3, \ r > h_1 - h.$$

Let $r \leq h_1 - h(< 0)$. If we prove that for some $\bar{C} > 0$ $\bar{k} > 0$ and $\bar{h}_0 > h_1$

$$(12^*) \qquad 1 \leq \bar{C}\, h^{\nu}L(h)(1 + |r|)^{\bar{k}}, \ h > \bar{h}_0$$

then the part of the proof in which we assume that L is non--decreasing, will be complete. For $\nu \geq 0$ (12*) holds trivially and for $\nu < 0$, (12*) follows from the inequality

$$h \leq |r| + h_1.$$

Suppose that L is a non-increasing function for $h > h_1$.

Let $r + h > h_1$ and $h > 3h_1$. For $0 > r \geq (h_1 - h)/2$ we have

$$1/6 \leq 1 + r/h \leq 1.$$

So, by (3) we have that for some $C_3 > 0$ and some $h_4 > 3h_1$

$$(13) \qquad (h+r)^{\nu}L(h+r) \leq C_3(h+r)^{\nu}L(h), \ h > h_4.$$

If $h_1 - h < r < (h_1 - h)/2$, $h > h_4$, then

$$h < h_1 + 2|r| \quad \text{and} \quad L(h)(1 + |r|)^{k_1} \geq C_4$$

for some $C_4 > 0$ and some $k_1 > 0$. Thus from

$$L(h+r) \leq L(h_1) \leq C_5L(h)(1 + |r|)^{k_1},$$

we obtain

$$(14)\quad (h+r)^{\nu}L(h+r) \leqslant C_5(h+r)^{\nu}L(h)(1+|r|)^{k_1},$$

$$h > h_4,\quad h_1 - h < r < (h_1 - h)/2.$$

For $r > 0$, $h > h_4$

$$(15)\quad (h+r)^{\nu}L(h+r) \leqslant (h+r)^{\nu}L(h).$$

From (11), (13), (14) and (15) it follows that for some $h_0 > 0$, $k > 0$ and $C > 0$ (12) holds.

Let $r \leqslant h_1 - h < 0$. If for some $\bar{C} > 0$, $\bar{k} > 0$ and $\bar{h}_0 > h_1$, (12*) holds, the proof will be complete. If $\nu > 0$, then (12*) follows from (2). If $\nu \leqslant 0$, then from

$$1 \leqslant C_5 h^{\nu}L(h)(1+h)^{\bar{k}},\ h > \bar{h}_0,$$

which holds for suitable $\bar{C} > 0$, $\bar{h}_0 > h_1$, $\bar{k} > 0$, and

$$h < |r| + h_1,$$

(12*) follows. The proof is complete.

Let us denote by $\Sigma_p^*(R)$ a subset of $\Sigma_p(R)$ whose elements are functions $c(h)$ which satisfy (P.2) and

$$\text{(P.1)*}\qquad \begin{array}{l}\text{For some } C > 0,\ k > 0 \text{ and } h_0 > 0\\ c(h+r) \leqslant C\, c(h)(1+r)^n,\quad h > h_0,\ r > 0.\end{array}$$

Now, we have

PROPOSITION 2*. *Let* $c(h)$ *be of the form (7) where* $\nu \in R$, $h_1 > 0$ *and* L *is a slowly varying function for* $h > h_1$. *Then* $c(h)$ *satisfies* (P.1)*.

PROOF. It follows from (9), (10) and (11).

3. SET $\Sigma_e(R)$

Similarly as for $\Sigma_p(R)$, we shall show that conditions (E.1) and (E.2) are satisfied for a wide subset of $\Sigma_e(R)$.

Using (2) and (6) one can easily prove the following proposition.

PROPOSITION 3. *Suppose that* c(h) *is a function from* $\Sigma(R)$ *which is continuous in some neighbourhood of* ∞, *and that (5) holds with* $S \neq 0$. *Then (E.2) holds for* c(h).

PROPOSITION 4. *Let* $c(h) \in \Sigma(R)$ *be of the form*

$$(16) \qquad c(h) = h^{\nu} L(h) \exp(\alpha |h|), \; h > h_1, \; c(h) = 1, h \leq h_1,$$

where $\nu \in R$, $h_1 > 0$, $r > 0$ *and* L(h) *is a monotonous slowly varying function for* $h > h_1$. *Then* (E.1) *holds for* c(h).

PROOF. Let us suppose that c(h) is of the form

$$(17) \qquad c(h) = \exp(\alpha |h|) \text{ for } h > h_1. \text{ and } c(h) = 1$$

for $h \leq h_1$, $h_1 > 0$, $\alpha \in R$. If $\alpha > 0$, then we have

$$(18) \qquad c(h+r) \leq c(h) \exp(\alpha |r|), \; h > h_1, \; r \in R.$$

Let $\alpha \leq 0$. Since for $r+h \geq h_1$, and $h > h_1$

$$|h+r| = h + |r| \quad \text{or} \quad |h+r| = h - |r|,$$

we have

(19) $c(h+r) \leq c(h)$ or $c(h+r) \leq c(h)\exp(\alpha|r|)$, $h > h_1$,

$r \geq h_1 - h$.

Let us prove that for some $h_0 > h_1$, $C > 0$, and some $k > 0$

(20) $1 \leq C\, c(h)\exp(k|r|)$, $h > h_0$, $r \leq h_1 - h$.

If $h_1 < h_0 < h < h_1 - r$, we have

$\exp(-\alpha|h|)\cdot\exp(-k|r|) \leq \exp(-\alpha|h_1| - \alpha|r|)\exp(-k|r|)$.

This implies, for $k = -\alpha$,

$1 \leq \exp(-\alpha|h_1|)c(h)\exp(-\alpha|r|)$, $h > h_0$, $r \leq h \;\; - h$,

i.e. we have proved (20).

Inequalities (18), (19) and (20) imply that the assertion of Proposition 4 holds for function $c(h)$ of the form (17).

Now, Proposition 3 implies the assertion of the proposition in a general case. The proof is complete.

We define $\Sigma_e^*(R)$ as a subset of $\Sigma(R)$ which elements satisfy (E.2) and

(E.1*) For some $C > 0$, $k > 0$, $h_0 > 0$

$c(h+r) \leq C\, c(h)\exp(k|r|)$, $h > h_0$, $r > 0$.

As in Proposition 2*, one can prove the following

PROPOSITION 4*. *Let* $c(h)$ *be of the form (16), where* $\nu \in R$, $h_0 > 0$ *and* L *is a slowly varying function for* $h > h_0$. *Then,* $c(h)$ *satisfies* (E.1)*.

At the end, let us remark that this paper enable us to formulate assertions from [2] in a more explicit form.

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REZIME

S-ASIMPTOTIKA TEMPERIRANIH I K_1'-DISTRIBUCIJA. DRUGI DEO. FUNKCIJE ZA POREDJENJE

U radu su ispitane funkcije c(h) za koje S-asimptotika može biti proširena sa prostora D(R) na S(R), odnosno sa prostora D(R) na prostor $K_1(R)$.