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A FIXED POINT THEOREM FOR MULTIVALUED CONDENSING MAPPINGS IN GENERAL TOPOLOGICAL VECTOR SPACES

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ABSTRACT

Using the results of Hadžić [5] and Jerofsky [8] a fixed point theorem of the Leray-Shauder type for condensing multivalued mappings in not necessarily locally convex spaces is proved.

1. There are many fixed point theorems for compact mappings in not necessarily locally convex topological vector spaces. An excellent survey about fixed point theorems in general topological vector spaces can be found in the book by Olga Hadžić [5]. However, the most important fixed point theorems for mappings of the condensing type (s. [1]) are unknown for nonlocally convex topological vector space. The

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first results in this direction Hadžić proved in [4], [5]. We shall prove in this paper such a general fixed point theorem for multivalued mappings of the condensing-type on domains which are locally convex, but not necessarily convex subsets of general topological vector spaces.

Let E be a real topological vector space and $K \subseteq E$. By $co(K)$, $\overline{co}(K)$ and $\bar{K}$ we shall denote the convex hull, the closed convex hull and the closed hull of K and by δK the boundary of K. We define $cc(K) =: \{A \subseteq K : A \neq \emptyset$, A is closed in K, A is convex$\}$,

$fu\,cc(E) := \{K \subseteq E : K = \bigcup_{i \in I} K_i$, I is finite, $K_i \in cc(E)$ for all $i \in I\}$.

Clearly we have $cc(E) \subset fucc(E)$.

2. First we shall give some notations and results about special admissible [5] sets.

DEFINITION 1. ([8]) *Let* E *be a topological vector space and* $K \subseteq E$. K *will be called a locally convex set, iff for all* $x \in K$ *there exists in* K *a base of neihbourhoods* $U(x)$ *of* x *with* $U(x) = W(x) \cap K$ *and* $W(x)$ *is a convex subset of* E.

It is clear, that for convex sets K Definition 1 is the same as the original definition in [9] (s. [5] too). Definition 1 implies easily that every subset K of a locally convex topological vector space is a locally convex set and that every subset of a locally convex set in a topological vector space is a locally convex subset again. Nontrivial examples of convex, locally convex subsets can be found in [5].

DEFINITION 2. [5]. *Let* E *be a topological vector space and* $K \subseteq E$. K *is a said to be of Zima's type, iff for every neighbourhood of* $0 \in E$ V *there exists a neighbourhood* U *of* 0, *such that* $co(U \cap (K-K)) \subseteq V$.

Hadžić proved (s. [5. p. 30]) that every convex subset of Zima's type is a locally convex set. By using Definition 1, we can generalize this result:

PROPOSITION 1. *Let* E *be a topological vector space. Then every subset* $K \subseteq E$ *of Zima's type is a locally convex set.*

P r o o f. As in [5, p. 30] we can show, that for all $x \in K$ and all neighbourhoods V of $0 \in E$, there exists a neighbourhood U of 0, such that we have $co((x+U) \cap K) \subseteq x+V$. The subset $W(x) := co((x+U) \cap K))$ is convex.

Since $W(x) \cap K \supseteq (U+x) \cap K$, the set $W(x) \cap K$ is in K a neighbourhood of x with $W(x) \cap K \subseteq x+V$. Therefore, K is locally convex.

Jerofsky [8] proved the following generalization of a result by Krauthausen [9] (s. [5] too).

PROPOSITION 2. *([8 ,Satz 1.5.3.]). Let* E *be a topological vector space and* $K \in fucc(E)$ *a locally convex subset of* E. *Then* K *is admissible.*

DEFINITION 3. ([2]). *Let* $(E, \| \ \|^*)$ *be a paranormed space and* $K \subseteq E$. *We say that* K *satisfies the Zima condition with the constant* r, *iff there exists* $r > 0$ *such that* $\| tx \|^* \leq rt \| x \|^*$ *for all* $t \in [0,1]$ *and all* $x \in K-K$.

We can prove easily ([5, p. 34]) that K is of Zima's type, if K satisfies the Zima condition (for nonconvex sets too).

Now we shall give, by using of an idea of Hadžić [3], an example of a nonconvex, admissible, locally convex subset of a nonlocally convex topological vector space. The space $S(0,1)$ of equivalence classes of finite measurable functions on $[0,1]$ with

$\| \hat{x} \|^* := \int_0^1 \frac{|x(s)|}{1+|x(s)|} d\mu \quad (\{x\} \in \hat{x})$ is a paranormed nonlocally convex space. Let $\lambda > 0$ be a real number and

$$K := \{\hat{x} \in S(0,1) : |x(s)| < \lambda (0 < s < 1)\} \cup \{\hat{x} \in S(0,1) : |x(s) - \lambda| < \lambda (0 < s < 1)\} .$$

We prove, that for K the Zima condition holds with $r = 1 + 3\lambda$.

Let $\hat{x} \in K$, $\hat{y} \in K$. For $|x(s)| < \lambda$, $|y(s)| < \lambda$ $(0 < s < 1)$ or for $|x(s) - \lambda| < \lambda$, $|y(s) - \lambda| < \lambda$ $(0 < s < 1)$, we have that

$$1 + |x(s) - y(s)| < 1 + 2\lambda <$$

$< (1 + 2\lambda)[1 + t|x(s) - y(s)|]$ $(0 < s < 1,\ 0 < t < 1)$ and, therefore, we obtain, as in [3], that $\| t(\hat{x}-\hat{y}) \|^* < (1+2\lambda)t \| \hat{x}-\hat{y} \|^*$.

For $|x(s)| < \lambda$ and $|y(s) - \lambda| < \lambda$ $(0 < s < 1)$, we have

$$1 + |x(s) - y(s)| < 1 + |x(s)| + |y(s) - \lambda| + \lambda < 1 + 3\lambda ,$$

and we obtain similarly as in the first case, that

$$\| t(\hat{x}-\hat{y}) \|^* < (1 + 3\lambda)\ t \| \hat{x}-\hat{y} \|^* .$$

Therefore for K there holds the Zima condition with $r = 1 + 3\lambda$ and K is of Zima's type and, therefore, locally convex.

Since $K \in \mathrm{fucc}(E)$, we can apply Proposition 2 and K is admissible.

3. Now we shall define the notion of the pseudocondensing mapping . The known nontrivial measures of noncompactness ϕ in locally convex spaces(s. [1]) are not measures of noncompactness in nonlocally convex topological vector spaces, since we have $\phi(M) \neq \phi(\mathrm{co}\, M)$ in general .

Therefore, we introduced in [7] the following notion .

DEFINITION 4. *Let E be a topological vector space,* $K \subseteq E$, A *a cone in a vector space, which defines the partial ordering* $\leq$ *and* γ *a system of subsets of* $\overline{\mathrm{co}}(K)$ *such that:*

$$M \in \gamma \Rightarrow (\overline{M} \in \gamma,\ \mathrm{co}M \in \gamma,\ M \cup \{u\} \in \gamma\ (u \in K),\ \ N \in \gamma\ (N \subseteq M)).$$

Let c *be a real number with* $c \geq 1$. *The function* $\phi : \gamma \to A$ *will* *be called a* c*-measure of noncompactness on* K*, iff the following conditions hold:*

(1) $\phi(M \cup \{u\}) = \phi(\bar{M}) = \phi(M) \geq \phi(N)$ $(M \in \gamma,\ N \subseteq M,\ u \in K)$

(2) $\phi(\mathrm{co}M) \leq c\phi(M)$ $(M \in \gamma)$.

If we have in (2) $c = 1$, then ϕ is said to be a measure of noncompactness on K (in this case we have in the fact $\phi(M) = \phi(\mathrm{co}M)$ $(M \in \gamma)$).

Now we shall give an example of a c-measure of noncompactness on a subset of a not necessarily locally convex space. Let (E,d) be a metric space and χ denote the well known Kuratowski function on E. This means that for all bounded sets $M \subseteq E$:

$$\chi(M) := \inf\{r > 0:\ M \text{ can be covered by a finite number of sets of diameter} \leq r\}.$$ Then we have

$\chi(M) = \chi(\bar{M}) = \chi(M \cup \{u\}) \geq \chi(N)$ $(N \subseteq M,\ u \in E)$ and M is precompact iff $\chi(M) = 0$. If E is a linear metric space, then we have $\chi(M+N) \leq \chi(M) + \chi(N)$ for all bounded subsets $M, N \subseteq E$, however, in general $\chi(\mathrm{co}M) \neq \chi(M)$. Therefore, χ is not a measure of noncompactness in general topological vector spaces. Using an idea of Hadžić [5], we can prove that χ is a c-measure of noncompactness in paranormed (s. [5]) spaces. Hadžić [5] proved that for a bounded, convex subset K of a paranormed space we have $\chi(\mathrm{co}M) \leq r^2\chi(M)$ $(M \subseteq K)$, if for K holds the Zima condition with the constant r. We can generalize this result, and so we obtain:

PROPOSITION 3. *Let* $(E, \|\ \|^*)$ *be a paranormed space and* K *a bounded subset of* E *with* $K \in \mathrm{fucc}(E)$. *We suppose that for* K *the Zima condition holds with the constant* $r \geq 1$. *Then the Kuratowski function* χ *is an* r^2*-measure of noncompactness on* K.

P r o o f. Let $M \subseteq K$ and $\varepsilon > 0$ with $\chi(M) < \varepsilon$. From the definition of χ it follows that there exists a cover $(B_j)_{j\in\{1,\dots,m\}}$ of M, such that diam $(B_j) < \varepsilon$ for all $j \in \{1,\dots,m\}$. Because $K \in \mathrm{fucc}(E)$, we can find closed, convex subsets K_k $(k = 1,\dots,p)$ with $K = \bigcup_{k=1}^{p} K_k$. Therefore, we obtain $M \subseteq \bigcup_{i=1}^{n} D_i$ with $n \leq m \cdot p$ and $D_i = K_k \cap B_j$ for some $k \in \{1,\dots,p\}$ and $j \in \{1,\dots,m\}$ $(i = 1,2,\dots,n)$. Then we have diam $(D_i) < \varepsilon$ and $\overline{\mathrm{co}} D_i \subseteq K_k \subseteq K$ for all $i \in \{1,2,\dots,n\}$. Now we can continue the proof with the cover $(D_i)_{i\in\{1,\dots,n\}}$, as the proof in [5, p. 57-60].

We cannot obtain the known fixed point theorems for χ-condensing mappings in general topological vector spaces.

Therefore, we introduced in [7] the following notion:

DEFINITION 5. *Let* E *be a topological vector space,* $M \neq \emptyset$, $K \neq \emptyset$, $M \subseteq K \subseteq E$, $F: M \to cc(K)$ *a (multivalued) upper semi-continuous mapping,* $c \geq 1$ *and* ϕ *a* c*-measure of noncompactnes on* K. *We call* F *a* ϕ*-pseudo-condensing mapping, iff the following implication holds:*

$[N \subseteq M,\ \phi(N) \leq c\phi(F(N)) \implies \overline{F(N)}$ *is compact.*$]$.

If ϕ *is a measure of noncompactness on* K *(which means* $c=1$*), then* F *will be called* ϕ*-condensing.*

Now we shall give a nontrivial example for an χ-pseudo-condensing mapping in a not necessarily locally convex space.

PROPOSITION 4. *Let* $(E, \|\ \|^*)$ *be a quasicomplete paranormed space,* $M \neq \emptyset$, $K \neq \emptyset$ *be bounded with* $M \subseteq K \subseteq E$ *and* $K \in \mathrm{fucc}(E)$ *and for* K *holds the Zima condition with the constant* $r \geq 1$. *Let* $F: M \to cc(K)$ *be a mapping with:*

(1) $F = F_1 + F_2$

(2) $F_1: M \to K$ *is a generalized* $\frac{1}{r^2}$ *-contraction.*

(This means, that there exists a real function $q(\alpha,\beta)$ *with* $0 < q(\alpha,\beta) < \frac{1}{r^2}$, *such that we have*

$$\| F_1x - F_1y\| ^* < q(\alpha,\beta) \| x - y\| ^* \text{ for all } x \in M,\ y \in M \text{ with } \alpha \leq \| x - y\| ^* \leq \beta).$$

(3) $F_2 : M \to cc(K)$ *is upper semicontinuous and* $\overline{F_2(M)}$ *is compact.*

Then F *is an* χ*-pseudo-condensing mapping.*

P r o o f. Because of Proposition 3 χ is an r^2-measure of noncompactness on K. We can show with known arguments (s. [11]) as in [7, Theorem 3], that for all $A \subseteq M$, which are not relatively compact, we have $\chi(F_1(A)) < \frac{1}{r^2}\chi(A)$. Since $F(A) \subseteq F_1(A) + F_2(A)$, we obtain $\chi(F(A)) \leq \chi(F_1(A)) + \chi(F_2(A)) < \frac{1}{r^2}\chi(A)$, because $\overline{F_2(A)}$ is compact. Therefore, condition $\chi(A) \leq r^2\chi(F(A))$ implies the compactness of $\bar{A}$, and the $\overline{F(A)}$ is compact, because F is upper semicontinuous and F(x) is comfor all $x \in M$.

4. Now we shall prove our fixed point theorem. Jerofsky [8, Folgerung 4.3.5] proved the following

THEOREM A. *Let* E *be a topological vector space and* K_o *an admissible subset of* E *with* $K_o \in fucc(E)$, *which is starshaped, relative to some* $u \in K_o$. *Let* $Y \subseteq K_o$ *be an in* K_o *closed neighbourhood of* u *in* K_o, *let* $F : Y \to cc(K_o)$ *be a compact mapping and*

$$x \notin t\,F(x) + (1-t)u \quad (x \in \delta_{K_o} Y,\ t \in (0,1)) .$$

Then there exists a fixed point for F.

Using this result, we can obtain a fixed point theorem for pseudo-condensing mappings.

THEOREM. *Let* E *be a topological vector space,* K *a locally convex subset of* E *such that* K *is starshaped, relative* *to some* $u \in K$ *and* $K \in \mathrm{fucc}(E)$.

Let $M \subseteq K$ *be an in* K *closed neighbourhood of* $u \in K$, $c \geq 1$ *and* ϕ *an* c-*measure of noncompactness on* K. *Let* $F: M \to cc(K)$ *be a* ϕ-*pseudo-condensing mapping with*

$$x \notin tF(x) + (1-t)u \quad (x \in \delta_K M,\ t \in (0,1)).$$

Then F *has a fixed point.*

P r o o f. Let $\Sigma := \{S \subseteq E : S = \overline{co}S,\ u \in S,\ F(M \cap S) \subseteq S\}$. Then $\Sigma \neq \emptyset$ and $S_o := (\bigcap_{S \in \Sigma} S) \in \Sigma$. Let $S_1 := \overline{co}(\{u\} \cup F(M \cap S_o))$.

Then we obtain $S_1 \subseteq S_o$ and $F(M \cap S_1) \subseteq F(M \cap S_o) \subseteq S_1$, $u \in S_1$, and therefore $S_1 \in \Sigma$. Then there must be $S_o \subseteq S_1$, and we obtain $S_o = \overline{co}(\{u\} \cup F(M \cap S_o))$, and then $M \cap S_o \subseteq \overline{co}(\{u\} \cup \cup F(M \cap S_o))$. The property of ϕ implies $\phi(M \cap S_o) \leq c\phi(F(M \cap S_o))$. Because F is ϕ-pseudo-condensing, $\overline{F(M \cap S_o)}$ is compact.

Let $Y := M \cap S_o$ and $F_o = F \mid Y$. We have $F_o(Y) \subseteq K \cap S_o = {}: K_o$. K_o is starshaped relative u and $K_o \in \mathrm{fucc}(E)$, since S_o is convex and $K \in \mathrm{fucc}(E)$. Y is an in K_o closed neighbourhood of $u \in K_o$. We have $\delta_{K_o} Y \subseteq \delta_K M$ and therefore $x \notin tF_o(x) + (1-t)u$ for all $t \in (0,1)$ and $x \in \delta_{K_o} Y$. Because K is locally convex, the subset K_o is locally convex too.

Since $K_o \in \mathrm{fucc}(E)$, we can apply Proposition 2, and K_o is admissible. Now we apply Theorem A, and the compact mapping $F_o : Y \to cc(K_o)$ has a fixed point $x \in F_o(x)$. Then x is a fixed point for F.

We remark that our theorem and Theorem A do not follow by using our theorem in [6] (s.[5],p.130) ,because it is unknown that K_o is an $\mathcal{R}$-admissible set.

The theorem generalizes many known fixed point theorems (s. [1], [5], [10]).

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REZIME

TEOREMA O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA KONDENZUJUĆA PRESLIKAVANJA U OPŠTIM VEKTORSKO TOPOLOŠKIM PROSTORIMA

Koristeći rezultate Hadžić [5] and Jerofsky [8] u ovom radu je dokazana teorema o nepokretnoj tački Leray--Schauderovog tipa za kondenzujuća preslikavanja u neobavezno lokalno konveksnim prostorima .