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THE METRIZATION OF PROBABILISTIC METRIC SPACES WITH APPLICATIONS

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ABSTRACT

The purpose of this paper is to study a metrization of a class of probabilistic metric spaces. As an application, we consider the existence of fixed points for some kinds of mappings in probabilistic metric spaces, and give some new fixed point theorems which generalize some recent results of [1], [2], [7], [8], [11], [12].

1. INTRODUCTION

The metrization of a probabilistic metric space is of the fundamental importance in the theory and applications of probabilistic metric spaces and has been considered by Schweizer, Sklar and Thorp [9], Moynihan and Schweizer [5], Hicks [3], Hicks and Sharma [4] and Radu [6].

In this paper we discuss the Hicks metric from [3] and

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as an application, in section 3. we consider the existence of fixed points for some kinds of mappings in Menger space $(S,\mathcal{F},\min)$. The obtained results are generalizations of fixed point theorems from [1], [2], [7], [8], [11], [12].

2. PRELIMINARIES

Throughout this paper, we denote $R = (-\infty, \infty)$, $\mathcal{D}$ the set of left-continuous distribution functions and H the function

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

DEFINITION 2.1. *A function* $T:[0,1]^2 \to [0,1]$ *is called to be a* t*-norm if for any* $a,b,c,d \in [0,1]$:

$$T(a,1) = a,\ T(a,b) = T(b,a),\ T(a,b) \geq T(c,d)\ (a \geq c, b \geq d)$$

and

$$T(a, T(b,c)) = T(T(a,b),c).$$

The notion of a probabilistic metric space is introduced by K. Menger. A special probabilistic metric space is the Menger space [10].

DEFINITION 2.2. *A Menger space is a triplet* $(E,\mathcal{F},T)$, *where* E *is a nonempty set,* T *is a* t*-norm and* $\mathcal{F}$ *is a mapping of* $E \times E$ *into* $\mathcal{D}$ *(we shall denote the distribution functions* $\mathcal{F}(x,y)$ *by* $F_{x,y}$, *and* $F_{x,y}(t)$ *will represent the value of* $F_{x,y}$ *at* $t \in R$) *satisfying the following conditions:*

(PM-1) $F_{x,y}(t) = H(t)$, *for every* $t>0$ *if and only if* $x=y$.

(PM-2) $F_{x,y}(0) = 0$, *for every* $(x,y) \in E \times E$.

(PM-3) $F_{x,y} = F_{y,x}$, *for every* $(x,y) \in E \times E$.

(PM-4) $F_{x,y}(t_1 + t_2) \geq T(F_{x,z}(t_1), F_{z,y}(t_2))$, *for every*

$x,y,z \in E$ *and every* $t_1,t_2 \in [0,\infty)$.

It is well-known [10] that if (E,F,T) is a Menger space with t-norm T satisfying $\sup_{a<1} T(a,a) = 1$ then (E,F,T) is a Hausdorff space in the topology τ (the so called (ε,λ)-topology), induced by the family of neighbourhoods $\{U_p(\varepsilon,\lambda): p \in E, \lambda > 0, \varepsilon>0\}$ where $U_p(\varepsilon,\lambda) = \{x \in E: F_{x,p}(\varepsilon) > 1-\lambda\}$. It is obvious that the families $\{U_p(\varepsilon,\varepsilon): p \in E, \varepsilon > 0\}$ and $\{U_p(\varepsilon,\lambda): p \in E, \varepsilon > 0, \lambda > 0\}$ are equivalent. With this topology the notions of the completeness and the continuity are introduced in the usual way [10].

In this paper we shall suppose that (E,F,T) is a Menger space such that $T(a,a) \geq a$, for every $a \in [0,1]$, which implies that $T = \min$.

In [3] Hicks defined the function $d: E \times E \to [0,1]$ in the following way:

$$d(x,y) = \begin{cases} \sup\{t \in (0,1): F_{x,y}(t) \leq 1-t\} \\ 0, \quad F_{x,y}(t) > 1-t, \text{ for every } t \in (0,\infty). \end{cases} \tag{2.1}$$

Then we have the following result [3].

THEOREM 2.1.

(i) *For any* $t \in (0,\infty)$: $d(x,y) < t$ *if and only if* $F_{x,y}(t) > 1-t$.

(ii) $F_{x,y}(d(x,y)) \leq 1-d(x,y)$, *for every* $(x,y) \in E \times E$.

(iii) d *is a metric on* E *which is compatible with topology* τ.

(iv) (E,F,T) *is* τ*-complete if and only if* E *is* d*-complete*.

PROOF. We shall prove only (ii) (for (i), (iii) and (iv) see Theorem 2.1 from [3]). If $d(x,y) = 0$ then the inequality in (ii) is satisfied. Suppose that $d(x,y) > 0$. Then for any sequence $\{t_n\} \subset \{t \in (0,1): F_{x,y}(t) \leq 1-t\}$ which increasingly converges to $d(x,y)$ we have that $F_{x,y}(d(x,y)) = \lim_{n\to\infty} F_{x,y}(t_n) \leq \lim_{n\to\infty}(1-t_n) = 1-d(x,y)$.

In what follows we give an example to illustrate what is the form of the metric d.

EXAMPLE. Let (E,ρ) be a complete metric space. We define the function $r: E \times E \to [0,1]$ as follows:

$$r(x,y) = \begin{cases} \rho(x,y), & \text{if } \rho(x,y) < 1 \\ 1, & \text{if } \rho(x,y) \geq 1. \end{cases} \tag{2.2}$$

It is obvious that r is a metric on E. Moreover, it follows that $\rho(x_n,x) \to 0 \iff r(x_n,x) \to 0$ and so the metric r and the metric ρ are equivalent to each other.

By virtue of the metric r we define a mapping $F: E\times E \to \mathcal{D}$:

$$F_{x,y}(t) = H(t-r(x,y)), \quad x,y \in E.$$

Taking t-norm $T = \min$ (i.e. $T(a,b) = \min\{a,b\}$, $\forall\, a,b \in [0,1]$) and noting Theorem 2 of [11] we know that $(E,\mathcal{F},\min)$ is a τ-complete Menger space, and the τ-convergence coincides with the r-convergence (therefore with the ρ-convergence too).

Now we define a metric d according to (2.2) on $(E,\mathcal{F},\min)$ as follows:

$$d(x,y) = \begin{cases} \sup\{t \in (0,1): F_{x,y}(t) \leq 1-t\}, \\ 0, \quad F_{x,y}(t) > 1-t, \ \forall\, t \in (0,\infty), \end{cases}$$

$$= \begin{cases} \sup\{t \in (0,1),\ H(t-r(x,y)) \leq 1-t\}, \\ 0, \quad H(t-r(x,y)) > 1-t, \ \forall\, t \in (0,\infty). \end{cases}$$

It is easy to check that $d(x,y) = r(x,y)$.

The example stated above shows that the metric d defined by (2.1) is a generalization of the metric r defined by (2.2) in probabilistic metric space.

3. APPLICATIONS TO FIXED POINT THEORY

In this section, we shall utilize the result obtained above to study the fixed point theorems for mappings in probabilistic metric space.

Throughout this section we shall assume that (E,F,T) is a τ-complete Menger space, where the t-norm T satisfies the following condition:

$$T(t,t) \geq t, \ \forall\, t \in [0,1].$$

Suppose that d is the metric defined by (2.1) and the function Φ satisfies the following condition (Φ):

(Φ) $\Phi: [0,\infty) \to [0,\infty)$ is strictly increasing, right continuous and $\Phi(t) < t$, $\forall\, t > 0$.

LEMMA 3.1. *Let* f, g *be two self-mappings on* (E,F,T), *and* K *a function from* $E \times E$ *into* $[0,\infty)$. *Suppose that* p *and* q *are two mappings from* $E \times E$ *into* Z *(the set of all positive integers). Then*

$$(3.1) \qquad F_{f^{p(x,y)}x,\, g^{q(x,y)}y}(\Phi(t)) > 1-\Phi(t), \ \forall t > K(x,y), \quad x,y \in E,$$

if and only if

$$(3.2) \qquad d(f^{p(x,y)}x,\ g^{q(x,y)}y) \leq \Phi(K(x,y))$$

PROOF. Necessity. Suppose that

$$F_{f^{p(x,y)}x,\ g^{q(x,y)}y}(\Phi(t)) > 1-\Phi(t), \quad \forall t > K(x,y),$$

$$x,y \in E.$$

Letting $\varepsilon = t-K(x,y)$ and noting the conclusion of Theorem 2.1 (i) we have

$$d(f^{p(x,y)}x,\ g^{q(x,y)}y) < \Phi(t) = \Phi(\varepsilon + K(x,y)).$$

Letting $\varepsilon \searrow 0$ and using the right continuity of Φ we have

$$d(f^{p(x,y)}x,\ g^{q(x,y)}y) \leq \Phi(K(x,y)).$$

Sufficiency. It follows from the strictly increasing property of Φ and the inequality (3.2) that

$$d(f^{p(x,y)}x,\ g^{q(x,y)}y) \leq \Phi(K(x,y)) < \Phi(t),\ \forall\ t > K(x,y).$$

In view of Theorem 2.1 (i) we know that (3.1) is true.

This completes the proof.

In the sequel, we denote $O_f(x;i,\infty) = \{f^n x\}_{n=i}^{\infty}$, $i = 0,1,2,\ldots$; $O_f(x,y;i,\infty) = O_f(x;i,\infty) \cup O_f(y;i,\infty)$, $i = 0,1,2,\ldots$, and $\delta(A) = \sup_{x,y \in A} d(x,y)$, where A is a subset of E.

THEOREM 3.2. *Let* f *be a* τ*-continuous self-mapping on* (E,F,T). *Suppose that there exist positive integers* p,q *such that for all* $x,y \in E$ *and all* $t > 0$, $t > K(x,y)$, *where*

$$K(x,y) = \delta(O_f(x,y;0,\infty)),$$

the following holds

$$F_{f^p x,\ f^q y}(\Phi(t)) > 1-\Phi(t).$$

Then there exists a unique fixed point of f *in* E, *and for any* $x_0 \in E$, *the iterative sequence* $\{f^n x_0\}$ τ*-converges (hence* d*-converges) to this fixed point.*

PROOF. By the assumptions of this Theorem and Lemma 3.1 for any $x,y \in E$ we have

$$d(f^p x,\ f^q y) \le \Phi(\delta(O_f(x,y;0,\infty))).$$

By Theorem 1.8.6 of [1] we know that there exists a unique fixed point of f and for any $x_0 \in E$ the iterative sequence $\{f^n x_0\}$ τ-converges (hence d-converges) to this point.

This ends the proof.

REMARK 1. As Theorem 1.8.7 of [1] pointed out , when p = q = 1 in Theorem 3.2 the continuity of f can be dropped.

REMARK 2. Taking $p = q = 1$, $\Phi(t) = \alpha t$, $\alpha \in (0,1)$ and

$$K(x,y) = d(x,y),$$

or

$$K(x,y) = \max\{d(x,y),\ d(x,fx),\ d(y,fy),\ d(x,fy),\ d(y,fx)\}$$

in Theorem 3.2 we know that Theorem 3.2 is a generalization of Banach fixed point theorem or Ćirić fixed point theorem (see [2]) in probabilistic metric spaces.

THEOREM 3.3. *Let* f *be a* τ*-continuous self-mapping on* (E,F,T). *Suppose that for any* $x \in E$ *there exists positive integer* p(x) *such that for any* $u,v \in O_f(x;0,\infty)$ *and any* $t > d(u,v)$ *the following holds*

$$F_{f^{p(x)}u,\ f^{p(x)}v}(\Phi(t)) > 1 - \Phi(t).$$

Then there exists a fixed point of f, *and for any* $x_0 \in E$ *the itera-*

tive sequence $\{f^n x_0\}$ *,τ-converges to some fixed point of* f *in* E.

PROOF. Taking $K(u,v) = d(u,v)$, $u,v \in O_f(x;0,\infty)$ it follows from the assumptions and Lemma 3.1 that for any $u,v \in O_f(x;0,\infty)$

$$d(f^{p(x)}u,\ f^{p(x)}v) \le \Phi(d(u,v)).$$

Therefore we have

$$\sup_{u,v\in O_f(x;0,\infty)} d(f^{p(x)}u,\ f^{p(x)}v) =$$

$$= \sup_{u,v\in O_f(x;p(x),\infty)} d(u,v) \le \sup_{u,v\in O_f(x;0,\infty)} \Phi(d(u,v)) \le$$

$$\le \Phi\Big(\sup_{u,v\in O_f(x;0,\infty)} d(u,v)\Big)$$

By Theorem 1.8.2 of [1] we know that the conclusion of Theorem 3.3 is true.

This completes the proof.

THEOREM 3.4. *Let* f *be a τ-continuous self-mapping on* (E,F,T). *Suppose that there exists positive integer* p *such that one of the following conditions is satisfied.*

(i) for any $x \in E$ *and any* $u,v \in O_f(x;0,\infty)$ *the following holds:*

$$F_{f^p u,\ f^p v}(\Phi(t)) > 1-\Phi(t),\ \forall\ t > d(u,v);$$

(ii) for any $x\in E$ *and any nonnegative integer* k *the following holds*

$$F_{f^p x,\ f^{p+k} x}(\Phi(t)) > 1-\Phi(t),\ \forall\ t > d(x,f^k x).$$

Then the conclusion of Theorem 3.3 remains true.

PROOF. If condition (i) is satisfied, then the conclusion of theorem follows from Theorem 3.3 immediately. If condition (ii) is satisfied letting $y = f^k x$, and $K(x,y) = d(x,f^k x)$ it follows from Lemma 3.1 that for any $x \in E$ and any nonnegative integer k

$$d(f^p x, f^{p+k} x) \leq \Phi(d(x,f^k x)).$$

Hence for any nonnegative integers r,s, $r \leq s$ we have

$$\begin{aligned} d(f^{p+r}x,\ f^{p+s}x) &= d(f^p f^r x,\ f^{p+s-r} f^r x) \\ &\leq \Phi(d(f^r x,\ f^{s-r} f^r x)) \\ &\leq \Phi(d(f^r x,\ f^s x)) \\ &\leq \Phi(\delta(O_f(x;0,\infty))). \end{aligned}$$

This shows that for any $x \in E$ we have

$$\delta(O_f(x;p,\infty)) = \sup_{r,s\geq 0} d(f^{p+r}x,\ f^{p+s}x) \leq \Phi(\delta(O_f(x;0,\infty))).$$

By Theorem 1.8.5 of [1] the conclusion of Theorem 3.4 is true.

THEOREM 3.5. *Let* f *be a self-mapping on* (E,F,T). *Suppose that for each* $x \in E$ *there exists a positive integer* n(x) *such that for all* $y \in E$ *and all* $t > 0$, $t > K(x,y)$, *where*

$$K(x,y) = \max\{d(x,y),\ d(x,\ f^{n(x)}x),\ d(x,\ f^{n(x)}y)\},$$

the following holds: $F_{f^{n(x)}x,\, f^{n(x)}y}(\lambda t) > 1-\lambda t$

where $\lambda \in (0,1)$. *Then* f *has a unique fixed point in* E *, and for any* $x_o \in E$ *the iterative sequence* $\{f^n x_o\}$ *τ-converges to this fixed point.*

Proof: Since we have that for every $x \in E$ there exists $n(x) \in N$ so that

$$d(f^{n(x)}x, f^{n(x)}y) < \lambda \max\{d(x,y), d(x,f^{n(x)}x), d(x,f^{n(x)}y)\},\quad y \in E$$

the conclusion of Theorem 3.5 follows from Theorem 1.5.3 of [1].

REMARK 3. Theorem 3.5 is a generalization of the results of Rhoades [8] and Singh [12] in probabilistic metric spaces.

REMARK 4. By virtue of Lemma 3.1 we can obtain some other fixed point theorems in probabilistic metric spaces. For simplicity we omit the statement here.

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REZIME

METRIZACIJA VEROVATNOSNOG METRIČKOG PROSTORA SA PRIMENOM

Cilj ovog rada je ispitivanje metrizacije jedne klase verovatnosnih metričkih prostora. Kao primena dato je nekoliko teorema o nepokretnoj tački koje uopštavaju nedavne rezultate [1], [2], [7], [8], [11], [12].