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ON DEFINING THE DISTRIBUTION $(x_+^r)_-^{-s}$

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ABSTRACT

A definition is given for the distribution $F(f(x))$, where $F(x)$ is a distribution and $f(x)$ is a locally summable function. The particular case $F(x) = x_-^{-s}$ and $f(x) = x_+^r$ is then considered.

In the following we let N be the neutrix, see van der Corput [1], having domain $N' = \{1,2,\ldots,n,\ldots\}$ and range N'' the real numbers, with negligible functions linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r n$ for $\lambda > 0$ and $r = 1,2,\ldots$, and all functions which converge to zero as n tends to infinity.

Thus if

$$f(n) = f_1(n) + f_2(n)$$

where f_1 is negligible and the limit as n tends to infinity of $f_2(n)$ exists, then the neutrix limit as n tends to infinity of

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f(n) exists and

$$\underset{n\to\infty}{\text{N-lim}}\, f(n) = \lim_{n\to\infty} f_2(n).$$

In particular if $f_1(n) = n^2 \ln n + n^3$ and $f_2(n) = n^{-1} + 2$, then

$$\underset{n\to\infty}{\text{N-lim}}\, f(n) = 2.$$

Now let ρ be a fixed infinitely differentiable function having the properties

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1.$

We define the function δ_n by $\delta_n(x) = n\rho(nx)$ for $n = 1,2,\ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence converging to the Dirac delta-function δ.

We now define the locally summable function x_+^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases}$$

we define the locally summable function $\ln x_+$ by

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0 \end{cases}$$

and we define the distribution x_+^{-1} by

$$x_+^{-1} = (\ln x_+)'.$$

The distribution x_+^λ for $\lambda < -1$ is now defined inductively by

$$x_+^\lambda = (\lambda + 1)^{-1}(x_+^{\lambda+1})'$$

and the distribution x_-^λ is defined by

$$x_-^\lambda = (-x)_+^\lambda$$

for all λ.

The following definition was given in [3].

DEFINITION. *Let* F *be a distribution and let* f *be a locally summable function. We say that the distribution* F(f(x)) *exists and is equal to* h *on the open interval* (a,b) *if*

$$\underset{n\to\infty}{\text{N-lim}} \int_{-\infty}^{\infty} F_n(f(x))\phi(x)dx = (h,\phi)$$

for all test functions ϕ *with compact support contained in* (a,b), *where*

$$F_n(x) = F(x)*\delta_n(x)$$

for $n = 1,2,\ldots$.

This definition was considered in [2] for the case where F is a derivative of δ and in [4] for the case where f is an infinitely differentiable function.

The following theorem was proved in [3].

THEOREM 1. *The distributions* $(x_-^\mu)_-^\lambda$ *and* $(x_+^\mu)_-^\lambda$ *exist and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ *and* λ, $\lambda\mu \neq -1, -2, \ldots$

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\,\delta^{(-\lambda\mu-1)}$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ *and* $\lambda\mu = -1, -2, \ldots$.

We now prove the following theorem.

THEOREM 2. *The distribution* $(x_+^r)_-^{-s}$ *exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}\, c(\rho)}{r(rs-1)!}\, \delta^{(rs-1)} \tag{1}$$

for r, s = 1,2,..., *where*

$$c(\rho) = \int_0^1 \ln t\rho(t)dt.$$

PROOF. We put

$$(x_-^{-s})_n = x_-^{-s} * \delta_n(x) = -\frac{1}{(s-1)!} \ln x_- * \delta_n^{(s)}(x)$$

for s = 1,2,... . Then

$$-(s-1)!(x_-^{-s})_n = \begin{cases} \int_{-1/n}^{1/n} \ln(t-x)\delta_n^{(s)}(t)dt, & x < -\frac{1}{n}, \\ \int_x^{1/n} \ln(t-x)\delta_n^{(s)}(t)dt, & |x| \leqslant \frac{1}{n}, \\ 0, & x > \frac{1}{n} \end{cases}$$

so that

$$-(s-1)!((x_+^r)_-^{-s})_n = \begin{cases} \int_{x^r}^{1/n} \ln(t-x^r)\delta_n^{(s)}(t)dt, & 0 \leqslant x \leqslant n^{-\frac{1}{r}}, \\ \int_0^{1/n} \ln t\delta_n^{(s)}(t)dt, & x < 0, \\ 0, & x > n^{-\frac{1}{r}} \end{cases}$$

for r, r = 1,2,... . It follows that $((x_+^r)_-^{-s})_n$ has its support contained in the interval $(-\infty, n^{-1/r})$.

We have

$$-(s-1)!\int_0^{n^{-1/r}} ((x_+^r)_-^{-s})_n x^i dx =$$

$$= \int_0^{n^{-1/r}} x^i \int_{x^r}^{1/n} \ln(t - x^r)\delta_n^{(s)}(t)dtdx =$$

$$= \int_0^{1/n} \delta_n^{(s)}(t) \int_0^{t^{1/r}} \ln(t - x^r)x^i dxdt =$$

$$= \frac{n^{s-(i+1)/r}}{r} \int_0^1 v^{(i+1)/r} \rho^{(s)}(v) \int_0^1 [\ln(v-uv) - \ln n] u^{(i+1)/r-1} dudv,$$

where the substitutions $x^r = tu$ and $nt = v$ have been made. It follows that

$$\int_0^{n^{-1/r}} ((x_+^r)_-^{-s})_n x^i dx$$

is negligible for $i \neq rs-1$. It also follows that when $i = rs$

$$\int_0^{n^{-1/r}} |((x_+^r)_-^{-s})_n x^{rs}| dx = O(n^{-1/r}).$$

When $i = rs-1$ we have

$$-r(s-1)! \int_0^{n^{-1/r}} ((x_+^r)_-^{-s})_n x^{rs-1} dx =$$

$$= \int_0^1 v^s \rho^{(s)}(v) \int_0^1 [\ln(v-uv) - \ln n] u^{s-1} dudv.$$

The part of the integral involving ln n is negligible and

$$\int_0^1 v^s \rho^{(s)}(v) \int_0^1 u^{s-1} \ln(v-uv)dudv =$$

$$= s^{-1}\int_0^1 v^s \ln v\, d\rho^{(s-1)}(v) +$$

$$+ s^{-1}\int_0^1 v^s\rho^{(s)}(v)dv \int_0^1 \ln(1-u)\ d(u^s -1) =$$

$$= \tfrac{1}{2}(-1)^s\ s^{-1}(s-1)! - \int_0^1 v^{s-1}\ \ln v\, d\rho^{(s-2)}(v) +$$

$$+ \tfrac{1}{2}(-1)^s(s-1)! \int_0^1 \frac{u^s - 1}{1 - u}\, du = \tfrac{1}{2}(-1)^s(s-1)! \sum_{j=1}^{s} j^{-1} +$$

$$+ (-1)^s(s-1)!c(\rho) - \tfrac{1}{2}(-1)^s(s-1)! \sum_{j=1}^{s} j^{-1} = (-1)^s(s-1)!c(\rho).$$

Thus

$$\underset{n\to\infty}{\text{N-lim}} \int_0^{n^{-1/r}} ((x_+^{\,r})_-^{\,-r})_n x^{rs-1}dx = -(-1)^s r^{-1} c(\rho).$$

Now let ϕ be an arbitrary test function with compact support contained in the interval (a,b), where we may suppose that $a < 0$ and $b > 1$. Then by Taylor's theorem

$$\phi(x) = \sum_{i=0}^{rs-1} \frac{x^i}{i!}\phi^{(i)}(0) + \frac{x^{rs}}{(rs)!}\phi^{(rs)}(\xi x)$$

where $0 \leqslant \xi \leqslant 1$.

It follows from what we have just proved that

$$\left|\int_0^b ((x_+^{\,r})_-^{\,-s})_n x^{rs}\phi^{(rs)}(\xi x)dx\right| \leqslant$$

$$\leqslant \sup_x\{|\phi^{(k)}(x)|\}\cdot \int_0^{n^{-1/r}} |((x_+^{\,r})_-^{\,-s})_n x^{rs}|dx \to 0$$

as n tends to infinity and so

$$\underset{n\to\infty}{\text{N-lim}} \int_0^b((x_+^{\,r})_-^{\,-s})_n\phi(x)dx =$$

$$= \underset{n\to\infty}{\text{N-lim}} \sum_{i=0}^{rs-1} \frac{\phi^{(i)}(0)}{i!} \int_0^{n^{-1/r}} ((x_+^{\,r})_-^{\,-s})_n x^i dx +$$

$$+ \lim_{n\to\infty} \frac{1}{(rs)!} \int_0^b ((x_+^r)_-^{-s})_n x^{rs} \phi^{(rs)}(\xi x)dx =$$

$$= - \frac{(-1)^s c(\rho)\phi^{(rs-1)}(0)}{r(rs-1)!} .$$

Further

$$\int_a^0 ((x_+^r)_-^{-s})_n \phi(x)dx = \int_0^{1/n} \ln t \delta_n^{(s)}(t)dt \int_a^0 \phi(x)dx$$

$$= n^s \int_0^1 \ln(v/n)\rho^{(s)}(v)dv \int_a^0 \phi(x)dx$$

and so

$$\operatorname{N-lim}_{n\to\infty} \int_a^0 ((x_+^r)_-^{-s})_n \phi(x)dx = 0 .$$

Thus

$$\operatorname{N-lim}_{n\to\infty} (((x_+^r)_-^{-s})_n, \phi) = \operatorname{N-lim}_{n\to\infty} \int_a^b ((x_+^r)_-^{-s})_n \phi(x)dx$$

$$= - \frac{(-1)^s c(\rho)\phi^{(rs-1)}(0)}{r(rs - 1)!} = \frac{(-1)^{rs+s} c(\rho)}{r(rs-1)!} (\delta^{(rs-1)}, \phi)$$

and equation (1) follows. This completes the proof of the theorem.

COROLLARY 1. *The distribution* $(x_-^r)_-^{-s}$ *exists and*

$$(x_-^r)_-^{-s} = \frac{(-1)^{s-1} c(\rho)}{r(rs-1)!} \delta^{(rs-1)}$$

for $r, s = 1,2,\ldots$.

PROOF. The result follows on replacing x by -x in equation (1).

COROLLARY 2. *The distributions* $(-x_+^r)_+^{-s}$ *and* $(-x_-^r)_+^{-s}$ *exist and*

$$(-x_+^r)_+^{-s} = (-1)^{rs-1}(-x_-^r)_+^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\,\delta^{(rs-1)}$$

for $r,\ s = 1,2,\ldots$.

PROOF. The results follow on noting that

$$(-x)_+^{-s} = x_-^{-s}$$

and so

$$(-x_+^r)_+^{-s} = (x_+^r)_-^{-s}, \quad (-x_-^r)_+^{-s} = (x_-^r)_-^{-s} .$$

THEOREM 3. *The distribution* $(|x|^r)_-^{-s}$ *exists and*

$$(|x|^r)_-^{-s} = \frac{2(-1)^{rs+s}c(\rho)}{r(rs-1)!}\,\delta^{(rs-1)} \tag{2}$$

for $r,\ s = 1,3,5,\ldots$.

PROOF. We have

$$-(s-1)!((|x|^r)_-^{-s})_n = \begin{cases} \displaystyle\int_{|x|^r}^{1/n} \ln(t-|x|^r)\delta_n^{(s)}(t)dt, & 0 \leqslant |x|^r \leqslant 1/n, \\ 0, & |x|^r > 1/n \end{cases} \tag{3}$$

for $r,\ s = 1,3,5,\ldots$. The function $((|x|^r)_-^{-s}$ is even and has its support contained in the interval $(-n^{-1/r}, n^{-1/r})$. It follows that

$$\int_{-n^{-1/r}}^{n^{-1/r}} ((|x|^r)_-^{-s})_n x^i dx = 0 \tag{4}$$

for odd i. For even i

(5) $$\int_{-n^{-1/r}}^{n^{-1/r}} ((|x|^r)_-^{-s})_n x^i dx = 2\int_0^{n^{-1/r}} ((|x|^r)_-^{-s})_n x^i dx$$

and so is negligible except when $i = rs-1$. Thus if ϕ is an arbitrary test function with compact support

$$\underset{n\to\infty}{\text{N-lim}}(((|x|^r)_-^{-s})_n,\phi) = 2\underset{n\to\infty}{\text{N-lim}}(((x_+{}^r)_-^{-s})_n,\phi)$$

and equation (2) follows. This completes the proof of the theorem.

COROLLARY. *The distribution* $(-|x|^r)_+^{-s}$ *exists and*

$$(-|x|^r)_+^{-s} = \frac{2(-1)^{rs+s}c(\rho)}{r(rs-1)!}\,\delta^{(rs-1)}$$

for r, $s = 1,3,5,\ldots$.

THEOREM 4. *The distribution* $(|x|^r)_-^{-s}$ *exists and*

(6) $$(|x|^r)_-^{-s} = 0$$

for r, $s = 1,2,\ldots$ *and* $rs \neq 1,3,5,\ldots$.

PROOF. Equations (3), (4) and (5) of course hold for r, $s = 1,2,\ldots$ and $i = 0,1,2,\ldots$. However, the critical case $i = rs-1$ is odd and so

$$\int_{-n^{-1/r}}^{n^{-1/r}} ((|x|^r)_-^{-s})_n x^i dx$$

is either zero or negligible for $i = 0,1,2,\ldots$ and $rs \neq 1,3,5,\ldots$. It follows that

$$\underset{n\to\infty}{\text{N-lim}}(((|x|^r)_-^{-s})_n,\phi) = 0 = (0,\phi)$$

for arbitrary test function ϕ and $rs \neq 1.3,5,\ldots$. Equation (6) follows. This completes the proof of the theorem.

COROLLARY 1. *The distribution* $(-|x|^r)_+^{-s}$ *exists and*

$$(-|x|^r)_+^{-s} = 0$$

for r, s = 1,2,... *and* rs ≠ 1,3,5,... .

COROLLARY 2. *The distribution* $(x^{2r})_-^{-s}$ *exists and*

$$(x^{2r})_-^{-s} = 0$$

for r, s = 1,2,... .

The result of corollary 2 was given in [5].

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REZIME

O DEFINICIJI DISTRIBUCIJE $(x_+^r)_-^{-s}$

Data je definicija distribucije F(f(x)), gde je F(x) distribucija i f(x) lokalno sumabilna funkcija. Ispitan je specijalan slučaj $F(x) = x_-^{-s}$ i $f(x) = x_+^r$.