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ON ALMOST CONTINUOUS MULTIFUNCTIONS

Valeriu Popa

Department of Mathematics, Higher Education Institute, 5500 Bacău, Romania

ABSTRACT

Almost continuous multifunctions were defined by us in [7] as a generalization of the univocal almost continuous applications, defined by Singal and Singal in [12]. Some properties of the almost continuous multifunctions are studied by Kovačević in [4] and Popa in [7] and [8]. The purpose of the present paper is to investigate some properties of these multifunctions and to obtain new characterizations of the lower almost continuous multifunctions.

DEFINITION 1. *Let* X *and* Y *be two topological spaces.*

(a) The multifunction $F:X \to Y$ *is upper almost continuous (u.a.c.) in* $x_0 \in X$ *if for every open set* $V \subset Y$ *with* $F(x_0) \subset V$ *there exists an open set* $U \subset X$ *containing* x_0 *that* $F(U) \subset \text{Int } \bar{V}$. *(*$\bar{V}$ *stands for the closure of* V*).*

(b) The multifunction $F:X \to Y$ *is lower almost continuous (l.a.c.) in* $x_0 \in X$, *if for every open set* $V \subset Y$ *with* $F(x_0) \cap V \neq \emptyset$, there *exists an open set* $U \subset X$ *containing* x_0,

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such that $F(x) \cap \operatorname{Int} \bar{V} \neq \emptyset$, $\forall x \in U$.

(c) The multifunction $F:X \to Y$ *is almost continuous (a.c.) in* $x_0 \in X$ *if it is both u.a.c. and l.a.c. in* x_0.

(d) The multifunction $F:X \to Y$ *is a.c. (u.a.c.; l.a.c.), if it has this property in each point* $x \in X$, [7].

DEFINITION 2. *A topological space is said to be almost compact if every open cover admits a finite subfamily, the closures of whose members cover the space* [10].

DEFINITION 3. *A topological space is said to be nearly compact if every open cover admits a finite subfamily, the interiors of the closures of whose members cover the space* [11].

REMARK 1. We have

compact => nearly compact => almost compact

but none of these implications is reversible [11].

THEOREM 1. *If the multifunction* $F:X \to Y$ *is almost continuous surjection and punctually compact and* X *is almost compact, then* Y *is almost compact.*

PROOF: Let $\mathcal{U} = \{U_j;\ j \in J\}$ be any open cover of Y. F(x) being compact, $\forall x \in X$, there exists a finite subfamily $\mathcal{U}' = \{U_{j_k}: k = 1,\dots,m\}$ of $\mathcal{U}$ such that

$$F(x) \subset \bigcup_{k=1}^{m} U_{j_k}.$$

Let

$$U_x = \bigcup_{k=1}^{m} U_{j_k}$$

The family $\{U_x: x \in X\}$ is then an open cover of Y and $\{\operatorname{Int} \bar{U}_x;\ x \in X\}$ is then a regular open cover of Y. F being an u.a.c.

surjection, then $\{F^+(\mathrm{Int}\,\bar{U}_x);\ x \in X\}$ is an open cover of X, according to Theorem 2.4, implication (1) => (4) from [7]. Since X is almost compact, therefore it has a finite subfamily $\{F^+(\mathrm{Int}\,\bar{U}_{x_i});\ i = 1,2,\dots,n\}$ such that

$$X = \bigcup_{i=1}^{n} \overline{F^+(\mathrm{Int}\,\bar{U}_{x_i})}.$$

F being l.a.c., according to Theorem 2.2, implication (1) => (5) from [7], we have

$$\overline{F^+(\mathrm{Int}\,\bar{U}_{x_i})} \subset \overline{F^+(\overline{\mathrm{Int}\,\bar{U}_{x_i}})} \subset F^+(\bar{U}_{x_i}) .$$

We have successively

$$Y = F(X) = F\Big(\bigcup_{i=1}^{n} \overline{F^+(\mathrm{Int}\,\bar{U}_{x_i})}\Big) \subset \bigcup_{i=1}^{n} F(F^+(\bar{U}_{x_i})) \subset$$

$$\subset \bigcup_{i=1}^{n} \overline{\bigcup_{k=1}^{m} U_{j_{i_k}}} = \bigcup_{i=1}^{n} \bigcup_{k=1}^{m} \bar{U}_{j_{i_k}} .$$

Thus Y is almost compact.

COROLLARY 1. *If the multifunction* F:X → Y *is an almost continuous surjection and punctually compact and* X *compact (nearly compact), then* Y *is almost compact.*

PROOF: Follows from Remark 1 and Theorem 1.

THEOREM 2. *If the multifunction* F:X → Y *is an u.a.c. surjection and punctually compact and* X *is compact, then* Y *is almost compact.*

PROOF. Let $U = \{U_i;\ i \in I\}$ be an open cover of Y. Because F(x) is compact, $\forall x \in X$, by a similar construction as in Theorem 1, the family $\{U_x;\ x \in X\}$ is an open cover of Y and the family $\{\mathrm{Int}\,\bar{U}_x;\ x \in X\}$ is a regular open cover of Y. F being an u.a.c. surjection, according to Theorem 2.4, implication (1)=>(4)

from [7], the family $U' = \{F^+(\text{Int } \bar{U}_x);\ x \in X\}$ is an open cover of X. Since X is compact, therefore it has a finite subfamily $\{F^+(\text{Int } \bar{U}_{x_i});\ i = 1,2,\dots,n\}$ such that

$$X = \bigcup_{i=1}^{n} F^+(\text{Int } \bar{U}_{x_i}) \subset \bigcup_{i=1}^{n} F^+(\bar{U}_{x_i}) .$$

We have successively

$$Y = F(X) = F(\bigcup_{i=1}^{n} F^+(\bar{U}_{x_i})) = \bigcup_{i=1}^{n} F(F^+(\bar{U}_{x_i})) \subset \bigcup_{i=1}^{n} \bar{U}_{x_i} =$$

$$= \bigcup_{i=1}^{n} (\overline{\bigcup_{k=1}^{m} U_{j_{i_k}}}) = \bigcup_{i=1}^{n} \bigcup_{k=1}^{m} \bar{U}_{i_{j_k}} .$$

Thus Y is almost compact.

COROLLARY 2. *An almost continuous image of an almost compact space is almost compact (Theorem 3.3; [11]).*

COROLLARY 3. *An almost continuous image of a compact space is an almost compact space (Corollary 3.2; [11]).*

DEFINITION 4. *We call a set net of the space X any multifunction of a directed set in the set X,* [1].

DEFINITION 5. *We call θ the upper limit of the $(A_n;\ n \in (D,>))$ set net, the set*

$$\theta - L_s A_n = \{x \in (X,T):\ \forall V_x \in T,\ \forall m \in D,\ \exists n \in D:$$

$$n > m \Rightarrow \bar{V}_x \cap A_n \neq \emptyset \}.$$

THEOREM 3. *If the multifunction $F:X \to Y$, where Y is T_3 space, is u.a.c. at the point $a \in X$, then for any net of elements from X $\{s_n;\ n \in (D,>)\}$ convergent to a, the net of sets $F(s_n)$ satisfies the relation*

$$\theta - L_s F(s_n) \subset \overline{F(a)}.$$

PROOF: Let F be u.a.c. at a and $y \in \theta - L_s(F(s_n))$ where $(s_n; n \in (D,>))$ is a net from X convergent to a. Let us suppose that $y \bar{\in} \overline{F(a)}$. Because Y is T_3 space, there exists an open set U and V containing y and $\overline{F(a)}$, respectively, such that $U \cap V = \emptyset$. From $\overline{F(a)} \subset V$, we have that $F(a) \subset V$. F being an u.a.c. at a, according to Theorem 2.3, implication (1) => (2) from [7], $a \in \text{Int}\,(F^+(\text{Int}\,\bar{V}))$. As $(s_n, n \in (D,>))$ converges to a, $\exists m \in D$ so that $\forall n > m$ having $s_n \in \text{Int}\,F^+(\text{Int}\,\bar{V}) \subset F^+(\text{Int}\,\bar{V})$ that is $F(s_n) \subset \text{Int}\,\bar{V}$. By $U \cap V = \emptyset$ it follows that $U \cap \text{Int}\bar{V} = \emptyset$, consequently $U \cap F(s_n) = \emptyset$, $\forall n > m$ and thus $y \bar{\in} \theta - L_s F(s_n)$, coming thus to a contradiction, thus $y \in \overline{F(a)}$. The theorem is proved.

DEFINITION 6. *A point* $x \in X$ *is said to be a δ-adherent of* $A \subset X$ *if* $V \cap A \neq \emptyset$ *for every regular open set* V *containing* x.

The set of δ-adherent points of A is called a δ-closure of A *and is denoted by* $\langle A \rangle$ [*13*].

DEFINITION 7. *A point* $x \in X$ *is the δ-interior of a set* $A \subset X$ *if there exists a regular open set V containing* x *and* $V \subset A$.

The set of δ-interior points of A *is called the δ-interior of* A, *and is denoted by δ-Int A.* [*3*].

DEFINITION 8. *A set* U *is called δ-open if* A = δ-Int A [3].

DEFINITION 9. *The net* $(x_n, n \in (D,>))$ *is δ-convergent to a* ($x_n \xrightarrow{\delta} a$), *if for every regular open set* $V \subset X$ *containing a, there exists* $n_0 \in (D,>)$ *such that* $\forall n > n_0 \Rightarrow x_n \in V$. [3].

DEFINITION 10. *Let* A *be a set of a topological space.* U *is a* δ - *neighbourhood of* A *which intersects* A, *if there exists a δ-open set* $V \subset X$, *such that* $V \subset U$ *and* $V \cap A \neq \emptyset$.

THEOREM 4. *For a multifunction* $F:X \to Y$, *the following are equivalent:*

1) F *is l.a.c. in* x_0.

2) For every $y_0 \in F(x_0)$ *and for every net* $(x_n, n \in (D,>))$ *convergent to* x_0, *there exists a subnet* $(z_b, b \in E)$ *of the net* $(x_n, n \in (D,>))$ *and a net* $(y_b, b \in E)$ *in* Y, *so that* $y_b \overset{\to}{\delta} y_0$ *and* $y_b \in F(z_b)$.

3) $x_0 \in \bar{A} \Rightarrow F(x_0) \subset \langle F(A) \rangle$ *for every subset* $A \subset X$.

4) $x_0 \in \overline{F^+(N)} \Rightarrow x_0 \in F^+(\langle N \rangle)$ *for every subset* $N \subset Y$.

PROOF. *(1)* $\Leftrightarrow$ *(2)*. It is similar to the proof of (1) $\Leftrightarrow$ (4) from Theorem 3 [9] placing in the implication (1) $\Rightarrow$ (4) of Theorem 3 from [9], the family V of regular open sets, and in the implication (4) $\Rightarrow$ (1) of Theorem 3 from [9], $\operatorname{Int} \bar{G}$ for $\bar{G}$.

(2) $\Rightarrow$ *(3)*. Let $x_0 \in \bar{A}$, then there exists a net $(x_n, n \in (D,>)) \in A$ such that $x_n \to x_0$. Let $y \in F(x_0)$. By hypothesis, there exists a subnet $(z_b, b \in E)$ of $(x_n, n \in (D,>))$ and a net $(y_b, b \in E)$ such that $y_b \in F(z_b) \subset A$ and $y_b \overset{\to}{\delta} y$, so $y \in \langle F(A) \rangle$ according to Theorem 2.4, implication (c) $\Rightarrow$ (a) from [3].

(3) $\Rightarrow$ *(1)*. Suppose that F is not l.a.c. in x_0. Then, there exists an open set $G \subset Y$ so that $G \cap F(x_0) \neq \emptyset$ and for any open set $U \subset X$ containing x_0, there is $x_U \in U$ for which $F(x_U) \cap \operatorname{Int} \bar{G} = \emptyset$. Let $M = \{x_U : U \in U_{x_0}\}$, where U_{x_0} is the family of open sets containing x_0, then $x_0 \in \bar{M}$, and by hypothesis $F(x_0) \subset \langle F(M) \rangle$. From the definition of the set M, $F(M) \cap \operatorname{Int} \bar{G} = \emptyset$. Since $F(x_0) \cap G \neq \emptyset$, $\exists\, z \in F(x_0) \cap G$, and so G is an open neighbourhood of z such that $\operatorname{Int} \bar{G} \cap F(M) = \emptyset$, which shows that $z \,\bar{\in}\, \langle F(M) \rangle$, and this contradicts the fact that $F(x_0) \subset \langle F(M) \rangle$.

(3) $\Rightarrow$ *(4)*. In (3), replacing A by $F^+(N)$ from $x_0 \in \overline{F^+(N)}$, we get $F(x_0) \subset \langle F(F^+(N)) \rangle \subset \langle N \rangle$, so $x_0 \in F^+(\langle N \rangle)$.

(4) $\Rightarrow$ *(3)*. Let $x_0 \in A$. In (4), let $N = F(A)$. Then $A \subset F^+(F(A)) = F^+(N)$, so $x_0 \in \overline{F^+(N)}$ and from hypothesis $F(x_0) \subset \langle N \rangle$, thus $F(x_0) \subset \langle F(A) \rangle$.

THEOREM 5. *For a multifunction* $F:X \to Y$ *the following are equivalent:*

1) F *is l.a.c.*

2) *For each point* x *of* X *and for each* δ*-neighbourhood* V *which intersects* $F(x)$, $F^{-}(V)$ *is a neighbourhood of* x.

3) *For each point* x *of* X *and for each* δ*-neighbourhood* V *which intersects* $F(x)$, *there is a neighbourhood* U *of* x *such that* $F(y) \cap V \neq \emptyset$, $\forall y \in U$.

4) $\overline{F(\bar{A})} \subset < F(A) >$ *for each subset* $A \subset X$.

5) $F^{+}(B) \subset F^{+}(< B >)$ *for each subset* $B \subset Y$.

6) *For each* δ*-closed set* $B \subset Y$, $F^{+}(B)$ *is a closed set.*

7) *For each* δ*-open set* $B \subset Y$, $F^{-}(B)$ *is an open set.*

8) $F^{-}(\delta\text{-Int } B) \subset \text{Int } F^{-}(B)$ *for each subset* $B \subset Y$.

PROOF. *(1) => (2).* Let $x \in X$, and $V \subset Y$ a δ-neighbourhood wich intersects $F(x)$, then there is a δ-open set $G \subset Y$ such that $G \subset V$ and $G \cap F(x) \neq \emptyset$. It is known that a set is δ-open in Y, if and only if G is the union of a family of regular open sets in Y. Then there is a regular open set $G_1 \subset G$ with $G_1 \cap F(x) \neq \emptyset$. According to Theorem 2.1, implication (1) => (4) from [7], there is an open set $U \subset X$ containing x, so that $F(y) \cap G_1 \neq \emptyset$, $\forall y \in U$, so $F(y) \cap G \neq \emptyset$, $\forall y \in U$, which implies $U \subset F^{-}(G)$. $G \subset V$ implies that $x \in U \subset F^{-}(G) \subset F^{-}(V)$ and thus $F^{-}(V)$ is a neighbourhood of x.

(2) => (3). Let $x \in X$ and $V \subset Y$ be a δ-neighbourhood which intersects $F(x)$. According to the hypothesis, $U = F^{-}(V)$ is a neighbourhood of x and $F(y) \cap V \neq \emptyset$, $\forall y \in U$.

(3) => (1). Let $x \in X$ and $V \subset Y$ be a regular open set such that $F(x) \cap V \neq \emptyset$. V being a regular open set is δ-open and thus V is a δ-neighbourhood which intersects $F(x)$ and according to the hypothesis, there is a neighbourhood U of x such that $F(y) \cap V \neq \emptyset$, $\forall y \in U$, so F is l.a.c. according to Theorem 2.1, implication (4) => (1) from [7].

(1) => (4). Let $y \in F(\bar{A})$, so there is $x \in \bar{A}$, so that $y \in F(x)$. According to Theorem 4, implication (1) => (3), $F(x) \subset < F(A) >$, so $y \in < F(A) >$.

(4) $\Rightarrow$ *(1)*. Let $x \in \bar{A}$, then $F(x) \subset F(\bar{A}) \subset \langle F(A) \rangle$ and according to Theorem 4, implication (3) $\Rightarrow$ (1), F is l.a.c. in x.

(1) $\Rightarrow$ *(5)*. Let $x \in \overline{F^+(N)}$, then according to Theorem 4, implication (1) $\Rightarrow$ (4), $F(x) \subset \langle N \rangle$, so $x \in F^+(\langle N \rangle)$.

(5) $\Rightarrow$ *(1)*. Let $x \in \overline{F^+(B)}$, then by hypothesis $x \in F^+(\langle B \rangle)$ and by Theorem 4, implication (4) $\Rightarrow$ (1), F is l.a.c. in x.

(5) $\Rightarrow$ *(6)*. Let B be a δ-closed set of Y, so $\langle B \rangle = B$. According to the hypothesis $\overline{F^+(B)} \subset F^+(\langle B \rangle) = F^+(B)$, thus $F^+(B)$ is closed.

(6) $\Rightarrow$ *(5)*. Let B be a subset of Y, then by Theorem 2.1 from [3], $\langle\langle B \rangle\rangle = \langle B \rangle$ and $\langle B \rangle$ is δ-closed. Then by hypothesis, $F^+(\langle B \rangle)$ is closed and $\overline{F^+(\langle B \rangle)} \subset F^+(\langle B \rangle)$. Since $\overline{F^+(B)} \subset \overline{F^+(\langle B \rangle)}$, then $\overline{F^+(B)} \subset F^+(\langle B \rangle)$.

(6) $\Leftrightarrow$ *(7)*. For if $B \subset Y$, then $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$.

(7) $\Rightarrow$ *(8)* δ-Int B is δ-open in Y and $F^-(\delta\text{-Int } B)$ is open in X. Since $F^-(\delta\text{-Int } B) \subset F^-(B)$, then $F^-(\delta\text{-Int } B) \subset \text{Int}F^-(B)$.

(8) $\Rightarrow$ *(1)* Let G be a regular open set of Y. Then G is δ-open and δ-Int G = G. Thus $F^-(G) \subset \text{Int } F^-(G)$ which shows that $F^-(G)$ is open in X and F is l.a.c. according to Theorem 2.2, implication (3) $\Rightarrow$ (1) from [7].

REMARK 2. Since the intersection of two regular open sets is a regular open set (Ex. 34(c); [2]), the regular open subsets of a space (Y,Q) may be used as a base for a topology Q' on Y.

COROLLARY 4. The multifunction $F:(X,T) \to (Y,Q)$ is l.a.c. if and only if the multifunction $F:(X,T) \to (Y,Q')$ is l.a.c.

PROOF: Follows from Remark 2 and the equivalence (1) $\Leftrightarrow$ (7) from Theorem 5.

REMARK 3: Some results from [3], [5], [6] and [12] are obtained for the univocal applications from Theorems 4, 5 and Corollary 4.

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REZIME

O GOTOVO NEPREKIDNIM VIŠEZNAČNIM FUNKCIJAMA

U radu [7] data je definicija pojma gotovo neprekidne višeznačne funkcije. Odgovarajući pojam za jednoznačna preslikavanja uveli su M.K. Singal i A.R. Singal u [12].

Neke osobine gotovo neprekidnih višeznačnih funkcija ispitivali su Kovačević u [4] i Popa u [7] i [8]. U ovom radu su ispitane neke osobine gotovo neprekidnih višeznačnih preslikavanja i dobijene nove karakterizacije od dole gotovo neprekidnih višeznačnih preslikavanja.