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HERMITE POLYNOMIALS AND THE PRODUCT OF GAUSSIAN MEASURES

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ABSTRACT

Let $\{\eta(t),\ t \geq 0\}$ be a Gaussian martingale and $H_p(\eta(t_1), \ldots, \eta(t_p))$ be the Hermite polynomial. In this paper it is proved that $\Delta^p H_p(\eta) = \prod_1^n [\eta(t_i + h_i) - \eta(t_i)]$ and this permits the defining of the multiple Ito-Rozanov stochastic integral by

$$\int_0^\infty \cdots \int_0^\infty \varphi(t_1, \ldots, t_p)\, d^p H(\eta).$$

1. Let $\{\xi(t),\ t \geq 0\}$ be a real Gaussian process and let $H_p(\xi(t_1), \ldots, \xi(t_p))$ be a Hermite polynomial of the degree p. (For an explicit expression of Hermite polynomials and some of its properties see, for example, [1]). Consider the increment of H_p:

$$(1) \qquad \Delta^p H_p(\xi) = H_p(\xi(t_1+h_1), \ldots, \xi(t_p+h_p)) -$$
$$- \sum_{i=1}^{n} H_p(\xi(t_1+h_1), \ldots, \xi(t_i), \ldots, \xi(t_p+h_p)) +$$

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$$\ldots (-1)^{p+1} H_p(\xi(t_1),\ldots,\xi(t_p)).$$

PROPOSITION.

(2) $$\Delta^p H_p(\xi) = \prod_{i=1}^{n} [\xi(t_i+h_i) - \xi(t_i)].$$

PROOF: We recall ([1]) the property of the Hermite polynomial of a sequence of, not necessarily different ,Gaussian variables $\xi_1,\ldots,\xi_p$:

$$\frac{\partial}{\partial \xi_k} H_p(\xi_1,\ldots,\xi_k,\ldots,\xi_p) =$$

$$= H_{p-1}(\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_p)$$

or

(3) $$H_p(\xi_1,\ldots,\xi_p) = \xi_k H_{p-1}(\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_p) +$$
$$+ P(\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_p).$$

Consider pairs of members on the right side of (1) differing in the first argument ($\xi(t_1+h_1)$ or $\xi(t_1)$) and identical in the other arguments. The members of such a pair have different signs. We have, using (3),

$$\Delta^p H_p(\xi) = H_p(\xi(t_1+h_1),\xi(t_2+h_2),\xi(t_3+h_3), \ldots$$
$$\ldots,\xi(t_p+h_p)) - H_p(\xi(t_1),\xi(t_2+h_2),\xi(t_3+h_3), \ldots$$
$$\ldots,\xi(t_p+h_p)) - H_p(\xi(t_1+h_1),\xi(t_2),\xi(t_3+h_3), \ldots$$
$$\ldots,\xi(t_p+h_p)) \ldots + H_p(\xi(t_1),\xi(t_2),\xi(t_3+h_3), \ldots$$
$$\ldots,\xi(t_p+h_p)) + \ldots (-1)^p H_p(\xi(t_1+h_1),\xi(t_2),\xi(t_3), \ldots$$
$$\ldots,\xi(t_p)) + (-1)^{p+1} H_p(\xi(t_1),\xi(t_2),\xi(t_3), \ldots$$
$$\ldots,\xi(t_p)) = [\xi(t_1+h_1) - \xi(t_1)] H_{p-1}(\xi(t_2+h_2), \ldots$$

$$\ldots,\xi(t_p+h_p)) - [\xi(t_1+h_1) - \xi(t_1)]\ H_{p-1}(\xi(t_2),\xi(t_3+h_3),$$

$$\ldots,\xi(t_p+h_p)) \ldots\xi(-1)^p[\xi(t_1+h_1) - \xi(t_1)]H_{p-1}(\xi(t_1),\ldots$$

$$\ldots,\xi(t_p)) = [\xi(t_1+h_1) - \xi(t_1)]\Delta^{p-1}H_{p-1}(\xi).$$

We get relation (2) by applying the same procedure on $\Delta^{p-1}H_{p-1}(\xi)$, and so on.

2. Now we shall consider a Gaussian martingale $\{\eta(t),\ t \geq 0\}$, $\eta(0) = 0$, with the continuous distribution function $F(t) = E\eta^2(t) = \|\eta(t)\|^2$, $t > 0$. If the intervals $(t_1+h_1,t_1),\ldots$ $\ldots,(t_p+h_p,t_p)$ are disjoint, we have for p-dimensional interval $\overset{p}{\underset{1}{X}}(t_i+h_i,t_i)$ that the stochastic measure $\prod_1^p[\eta(t_i+h_i)-\eta(t_i)]$ is equal $\Delta^pH_p(\eta)$, with $\|\Delta^pH_p(\eta)\|^2 = \prod_1^n[F(t_i+h_i) - F(t_i)]$.

In such a way we define the orthogonal stochastic measure $d^pH_p(\eta)$ on the Borel subset of

$$\overset{n}{\underset{1}{X}}[0,\infty) \setminus \bigcup_{i\neq j}\{u_i = u_j\}. \tag{4}$$

3. A multiple stochastic integral in the sense of Ito-Rozanov [3]

$$I_p(\varphi) = \int_0^\infty \ldots \int_0^\infty \varphi(u_1,\ldots,u_p)d\eta(u_1)\ldots d\eta(u_p),$$

$$(\|I_p(\varphi)\|^2 = \int_0^\infty \ldots \int_0^\infty \varphi^2(u_1,\ldots,u_p)dF(u_1)\ldots dF(u_p)<\infty),$$

is assumed to be the integral one the set defined by (4). It follows that $EI_p(\varphi) = 0$. Using our result, we have

$$I_p(\varphi) = \int_0^\infty \ldots \int_0^\infty \varphi(u_1,\ldots,u_p)d^pH_p(\eta). \tag{5}$$

Remark that relation (5) is used, without the proof, in [2] for $\{\eta(t)\}$ being the Wiener process.

Let $H_p(\eta)$ be the linear closure of Hermite polynomials $H_p(\eta(u_1),\ldots,\eta(u_p))$, $u_i \geq 0$. It is proved in [3] that

$$H_p(\eta) = \{I_p(\varphi),\ \varphi \in L_2\}.$$

This results follows immediately from relation (5), obviously,

$$\{I_p(\varphi)\} \subset H_p(\eta).$$

Using

$$\varphi_0(u_1,\ldots,u_i) = \begin{cases} 1, & u_i \leq t_i \\ 0, & \text{otherwise} \end{cases},$$

we have

$$H_p(\eta(t_1),\ldots,\eta(t_p)) = \int_0^\infty \ldots \int_0^\infty \varphi_0(u_1,\ldots,u_p)d^pH_p(\eta)$$

or

$$H_p(\eta) \subseteq \{I_p(\varphi)\}.$$

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REZIME

ERMITOVI POLINOMI I PROIZVOD GAUSOVIH MERA

Neka je $\{\eta(t),\ t \geq 0\}$ Gausov martingal i $H_p(\eta(t_1),$

..., $\eta(t_p)$) Ermitov polinom. U radu se dokazuje da je $\Delta^p H_p(\eta) = \prod_1^n [\eta(t_i+h_i) - \eta(t_i)]$ i to dopušta da se višestruki stohastički integral Ito-Rozanova definiše sa

$$\int_0^\infty \cdots \int_0^\infty \varphi(u_1,\ldots,u_p)\, d^p H_p(\eta).$$