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ON THE EQUIVALENCE OF DISTRIBUTIONS AT INFINITY

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ABSTRACT

The "asymptotic behaviour" of distributions, which naturally generalizes the "equivalence at infinity" used in [5], is defined. Its properties and relations to some other "asymptotics" of distributions or generalized functions are analyzed.

1. INTRODUCTION

The equivalence of a distribution at infinity with a function of the form x^p or $x^p \cdot \ln x$ for real p, which is not a negative integer, was used in a number of papers, for instance in [5] and [2], though it seems that this notion appeared in [8]. In this paper we shall define the equivalence at infinity of a distribution with a regularly varying function of the order $p \notin Z_-$, Z_- being the set of negative integers. After proving some properties, we shall compare it with the "asymptotic behaviour"

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of generalized functions used in [6] and with the quasiasymptotic behaviour of distributions analyzed in [3].

Throughout the paper, ρ will stand for a regularly varying function at infinity such that $\rho(x) = x^p L(x)$, $x > 0$, and $L:(0,\infty) \to R$ will always denote a locally integrable function on $(0,\infty)$ which is slowly varying at infinity, i.e. satisfies the condition

$$\text{(1)} \qquad \lim_{x\to\infty} L(tx)/L(x) = 1 \text{ for every } t>0 .$$

Similarly as in [7] it is supposed that L is of the same sign and nonzero in some neighbourhood of infinity.

Furthermore, D stands for the distributional derivative, $H_a(x)$ denotes the function which is zero for $x < a$ and equal to 1 for $x > a$ and the sign "$*$" the usual convolution between functions or distributions. S'_+ denotes the space of tempered distributions with supports in $[0,\infty)$. Finally, p will always denote a real number which is not a negative integer, and

$$\text{(2)} \qquad \begin{cases} C_{p,n} = \dfrac{1}{(p+1)(p+2)\dots(p+n)} & \text{for } n \in \mathbf{N}, \text{ and} \\ C_{p,0} = 1. \end{cases}$$

2. EQUIVALENCE AT INFINITY

We shall first rewrite a definition from [5].

DEFINITION 1. *The distribution* T *is equivalent at infinity with* Cx^p, $C \neq 0$, *if there exists* $n \in N_0$, $n + p > 0$, *a real number* $a > 1$ *and a continuous function* F *on* R *such that* $T = D^n F(x)$ *on* $[a,\infty)$ *and*

$$\text{(3)} \qquad F(x) \sim C C_{p,n} x^{p+n} \text{ as } x \to \infty$$

in the ordinary sense ($C_{p,n}$ *from* (2)). *It is supposed that*

$p \notin Z_-$ *if* $n > 0$.

In a later paper Lavoine and Misra defined the equivalence at infinity with $Cx^p \ln x$ for $p > -1$ and $C \neq 0$. It is natural to replace the logarithm with an arbitrary slowly varying function at infinity, and, if possible, omit the assumption on p. We shall do that in Definition 2 for real p which is not a negative integer; however the case $p \in Z_-$ cannot be handled in this manner, so we shall have to omit it.

DEFINITION 2. *The distribution* T *is equivalent at infinity with a regularly varying function* $\rho(x) = x^p L(x)$, $a > 1$, *if there exist* $n \in N_0$, $n+p > 0$, $a > 1$ *and a continuous function* F *on* R *such that* $T = D^n F(x)$ *on* (a,∞) *and*

$$F(x) \sim C_{p,n} x^{p+n} L(x) \text{ as } x \to \infty \tag{4}$$

in the ordinary sense. We then write $T \overset{E}{\sim} \rho(x)$ as $x \to \infty$.

Let us remark first that in Definition 2, we have omitted the equality of distributions on a set which is not open, namely $[a,\infty)$, used in the previous one. Also the constant $C \neq 0$ from (3) is now included in the slowly varying function L. Now, our task is to prove the correctness of Definition 2. The following lemma from [7] p. 86 shows that the increase of n does not change the equivalence at infinity:

LEMMA 1. *Let* F *be a positive locally integrable function on some interval* $[a,\infty)$, $a > 0$. *Then* $F(x) \sim x^p L(x)$ *as* $x \to \infty$ *implies*

$$\int_a^x F(t)dt \sim \frac{x^{p+1}}{p+1} L(x)$$

as $x \to \infty$, *provided that* $p > -1$.

It is obvious that the choice of $C_{p,n}$ in (2) was determined by this lemma. Next, we shall prove the uniqueness of the equivalence at infinity in a natural asymptotic sense.

THEOREM 1. *Let $T \in D'$ be equivalent at infinity with two regularly varying functions $\rho_i(x) = x^{p_i} L_i(x)$, where $p_i \notin Z_-$ and L_i are slowly varying function at infinity as indicated in the introduction,* $i = 1,2$. *Then* $p_1 = p_2$ *and*

$$\lim_{x \to \infty} \frac{L_1(x)}{L_2(x)} = 1.$$

PROOF. Let n_i, a_i and F_i $(n_i > -p_i)$, $i = 1,2$, be as in Definition 2, i.e. $T = D^{n_i} F_i(x)$ on (a_i,∞) and $F_i(x) \sim C_{p_i,n_i} \cdot x^{p_i+n_i} L_i(x)$ as $x \to \infty$. Let $a: = \max(a_1,a_2)$ and let us suppose that $n_1 \geqslant n_2$. Furthermore, let

$$G_1(x) = H_a(x) \int_a^x F_2(t)dt,$$

$$G_m(x) = H_a(x) \int_a^x G_{m-1}(t)dt, \quad m = 2,3,\ldots$$

and $x \in \mathbf{R}$. One can check easily that $T = D^{n_1}(G_{n_1-n_2}(x))$ on (a,∞) (for $n_1 = n_2$ one can take $G_0(x) = H_a(x)F_2(x)$). In view of Lemma 1, we have

$$G_{n_1-n_2}(x) \sim C_{p_2,n_1} x^{p_2+(n_1-n_2)+n_2} L_2(x)$$

as $x \to \infty$. Now $G_{n_1-n_2}(x)$ and $F_1(x)$ can differ on (a,∞) only by a polynomial of a degree less or equal to n_1-1, and this is possible only if $p_1 = p_2$ (since they are not negative integers) and if $L_1(x) \sim L_2(x)$ as $x \to \infty$.

It is natural to ask about the relationship between the classical asymptotic behaviour of a locally integrable function (which can be observed as regular distributions in some neighbourhood of infinity) as $x \to \infty$, and the equivalence at infinity.

Let us remember that in [6] such a definition of the asymptotic behaviour of distributions is given. Namely, if T is a distribution and h a locally integrable function in some interval (a,∞), then $T \sim h(x)$ as $x \to \infty$ in the sense of [6] means that in some neighbourhood of infinity T is defined with a locally integrable function, say f, and $f(x) \sim h(x)$ as $x \to \infty$ in the ordinary sense. For regularly varying function of order $p \notin Z_-$ the equivalence at infinity is a more general notion, as follows from

THEOREM 2. *Let* $T \in D'$ *be of the form* $T = B + f(x)$ *where* $B \in D'$ *has its support in* $(-\infty, a)$ *and* f *be a continuous function with support in* $(a-1,\infty)$, $a > 1$. *If* $f(x) \sim x^pL(x)$ *as* $x \to \infty$ *for some* $p \notin Z_-$ *and some slowly varying function* L *at infinity, then* $T \overset{E}{\sim} x^pL(x)$ *as* $x \to \infty$.

PROOF. The statement is trivial for $p > -1$. Let, now, $p = -(m+q)$ for $m \in N$ and $0 < q < 1$. Then by an analogue to Lemma 1 (see [7], p. 87) the function

$$G(x) := \int_a^x \int_{x_m}^\infty \cdots \int_{x_1}^\infty f(t)\,dt\,dx_1 \ldots dx_m,\quad x_1 \geq a, \ldots, x_m \geq a,\ x \geq a,$$

behaves like

$$-\frac{x^{p+m+1}}{(p+1)(p+2)\ldots(p+m+1)} L(x) = C_{p,m+1}x^{1-q} L(x) \text{ as } x \to \infty.$$

Observing that $D^{m+1}G(x) = f(x) = T$ on (a,∞), we finish the proof.

Let us remark that this theorem also gives a sufficient condition for equivalence at infinity. Let us prove now a necessary condition for the equivalence at infinity of a distribution with some regularly varying function. For that purpose, let us suppose additionally, that function L is both slowly varying at zero and infinity. Let $(x^pL(x))_+$ denote the following distribution from S' (the number $a > 1$ is chosen so that L is of the same sign on (a,∞)):

(5.a) $$\langle (x^pL(x))_+, \varphi \rangle := \int_0^\infty x^pL(x)\varphi(x)dx \text{ if } p > -1 \text{ and } \varphi \in S;$$

(5.b) $$< (x^pL(x))_+,\varphi >: = \int_0^\infty x^pL(x)(\varphi(x) - \varphi(0) - \dots \dots - \frac{x^{m-1}}{(m-1)!}\varphi^{(m-1)}(0))dx$$

if $-(m+1) < p < -m$, $m \in N$ and $\varphi \in S$ (see [4]). Since $(x^pL(x))_+ = x^pL(x)$ on (a,∞), obviously $(x^pL(x))_+ \overset{E}{\sim} x^pL(x)$ as $x \to \infty$.

We have come to

THEOREM 3. *Let the distribution* T *be equivalent at infinity with a regularly varying function* $\rho(x) = x^pL(x)$. *Then there exists a number* $b > 1$, *such that the distribution* R *defined by* $R: = T - (x^pL(x))_+$ *has the property*

(6) $$\lim_{k\to\infty} < R(x), \frac{\varphi(x/k)}{k\rho(k)} > = 0$$

for every $\varphi \in D$ *with the support in* (b,∞).

In order to prove this theorem, we shall prove two lemmas first. If $n \in N$ and $S \in S'_+$, then $H_0^{*n}*S(x)$ denotes the iterated convolution $\underbrace{H * H * \dots H}_{n\ times} * S(x)$ which is again in S'_+.

LEMMA 2. *Let* $n \in N$. *Then the iterated convolution* $H^{*n}*(x^pL(x))_+$ *is a tempered distribution which is equivalent at infinity with the regularly varying function* $C_{p,n}x^{n+p}L(x)$. *Moreover, there exists a locally integrable function* K *on* R *which is slowly varying at zero and at infinity, and satisfies the asymptotic behaviour*

(7) $$K(x) \sim C_{p,n}L(x) \text{ as } x \to \infty$$

such that $H_0^{*n}* (x^pL(x))_+ = (x^{p+n}K(x))_+$.

PROOF. First let $p > -1$. Then we have for $\varphi \in S$

$$< H_0*(x^pL(x))_+,\varphi >= \int_0^\infty x^pL(x)(\int_x^\infty \varphi(y)dy)dx = <(\int_0^x y^pL(y)dy)_+,\varphi(x)>$$

where the distribution $(\int_0^x y^p L(y)dy)_+$ is defined like the one in (5.a). Using Lemma 1, we get the statement for n = 1; the choice of K is obvious. For arbitrary n and p > -1, the proof follows by induction.

Now let $-(m+1) < p < -m$, for some natural number $m > 1$. Then

$$< H_0*(x^p L(x))_+,\varphi> = \int_0^\infty x^p L(x)(- \int_0^x \varphi(y)dy + \sum_{j=1}^{m-1} \frac{x^j}{j!} \varphi^{(j-1)}(0))\, dx =: < S,\varphi >,$$

where S is a functional to be analyzed. For that purpose, we shall observe the function $\sigma(x) = - \int_x^\infty y^p L(y)dy$ for $x > 0$; since $p < -1$, this function is well defined. First of all, it is a regularly varying function at infinity (see. [7], p. 87), namely

(8.a) $$\sigma(x) \sim \frac{x^{p+1}}{p+1} L(x) \text{ as } x \to \infty.$$

Next, σ is a regularly varying function at zero, too. In fact, we have

(8.b) $$\sigma(x) \sim \frac{x^{p+1}}{p+1} L(x) \text{ as } x \to 0+.$$

by transferring the statement from zero to infinity. A short calculation shows that in view of (8.b) and (5.b) the distribution S can be observed as a regularization of the locally integrable function σ on $(0,\infty)$. Hence, we can write

$$< H_0* (x^p L(x))_+,\varphi > = < S,\varphi > = < -(\int_x^\infty y^p L(y)dy)_+,\varphi > ,$$

so by (8.a) and Theorem 1 the distribution S is equivalent at infinity with the regularly varying function $x^{p+1}L(x)/(p+1)$. Thus we have proved the Lemma for n = 1 and $p < -2$; the choice of K and the remaining cases are similar to this one and are omitted.

LEMMA 3. *Let* f *be a locally integrable function on* $[b,\infty)$, $b > 0$, *such that* $f(x) = o(x^q L(x))$ *as* $x \to \infty$ *for some* $q > 0$ *and some slowly varying function* L *at infinity, which is also locally integrable on* $[b,\infty)$. *Then*

$$\int_b^\infty f(kx)g(x)dx = o(k^q L(k)) \text{ as } k \to \infty$$

for every locally integrable function g *on* $[b,\infty)$ *such that*

$$\int_b^\infty x^{q+r}|g(x)|dx < \infty$$

for some $r > 0$.

PROOF. By supposition for a given $\varepsilon > 0$, we can find a number $M = M(\varepsilon) > b$ such that the function $h(x) := f(x)/(x^q L(x))$ has the property $|h(x)| < \varepsilon$ for $x \geq M$. We take $k > M/b$ and obtain

$$\left|\int_b^\infty f(kx)g(x)dx\right| \leq \varepsilon\, k^q \int_b^\infty x^q L(kx)|g(x)|dx.$$

By [1], Theorem 2, we have

$$\int_b^\infty x^q L(kx)|g(x)|dx \sim L(k) \int_b^\infty x^q |g(x)|dx$$

or

$$\left|\int_b^\infty f(kx)g(x)dx\right| \leq C\, \varepsilon\, (k^q L(k)) \int_b^\infty x^q |g(x)|dx$$

for some $C > 0$ (which does not depend on ε or g). So, for a given ε we can find a $k_0 = k_0(\varepsilon)$ such that for $k \geq k_0$ we have

$$\frac{1}{k^q L(k)} \left|\int_b^\infty f(kx)g(x)dx\right| \leq C\, \varepsilon \int_b^\infty x^q |g(x)|dx\ .$$

PROOF OF THEOREM 3. Let $T \overset{E}{\sim} \rho(x)$ as $x \to \infty$ and let $n > -p$, F and $a > 1$ be as announced in Definition 2. This means that

$$F(x) = C_{p,n} x^n \rho(x) + f(x), \quad x > a,$$

and $f(x) = o(x^n \rho(x))$ as $x \to \infty$. Taking K from Lemma 2, observing that the function $g(x) := C_{p,n}\, x^{p+n}L(x) - x^{p+n}K(x)$ is also $o(x^n\rho(x))$ as $x \to \infty$, we get

$$< R(x), \frac{\varphi(x/k)}{k\rho(k)} > = < D^n F(x) - D^n(H_0^{*n} * (x^p L(x))_+), \frac{\varphi(x/k)}{k\rho(k)} >$$

$$= < D^n(f(x) + g(x)), \frac{\varphi(x/k)}{k\rho(k)} > =$$

$$= < \frac{(-1)^n}{k^{n+p}L(k)} \int_a^\infty (f(kx) + g(kx))\varphi(x)dx >.$$

Using, new, Lemma 3 we get the statement for $b := a$.

A condition analogous to (6) appeared in [5]. Unfortunately, this is not sufficient for the equivalence at infinity of a distribution T with a regularly varying function. For instance, if $T = e^{-t}$ on the interval $(1,\infty)$, then (6) holds for every regularly varying function $\rho(x) = x^p L(x)$, $p < 0$ and $p \in Z_-$, but, obviously, T is not equivalent at infinity with any regularly varying function. In fact, such distributions like T "tend" to zero faster then any regularly varying function. In view of that it is reasonable to give the following

DEFINITION 3. *A distribution* T *tends to zero faster than any degree of* x *at infinity, if there exist* $n_0 \in N$ *and* $a > 1$ *such that for every* $n \in N$, $n \geq n_0$, *there exists a continuous function* F_n *on* R *with the properties* $T = D^n F_n(x)$ *on* (a,∞) *and* $\lim_{x\to\infty} x^m F_n(x) = 0$ *for every* $m \in N$. *We then write* $T \overset{E}{\sim} 0$.

It is obvious that $T \in E'$, or more generally $T = 0$, on some interval (b,∞) implies $T \overset{E}{\sim} 0$. Similarly as Theorem 2, one can prove

THEOREM 4. *Let* $T \in D'$ *be of the form* $T = B + f(x)$, *where* $B \in D'$ *has its support in* $(-\infty,a)$ *and* f *be a continuous function with a support in* $(a-1,\infty)$, $a > 1$. *If function* f *satisfies the condition* $\lim_{x\to\infty} x^m f(x) = 0$ *for every* $m \in N$, *then* $T \overset{E}{\sim} 0$.

At the and of this section, we shall give a necessary and sufficient condition for the equivalence at infinity of a distribution with a regularly varying function or for $T \overset{E}{\sim} 0$. It will show that the equivalence at infinity is a local property of distributions, differing from the quasiasymptotic behaviour of distributions; the relation of these two asymptotics will be analyzed in Section 3.

THEOREM 5. *Let* $T = B + S$, *where* B *and* S *are distributions with supports in* $(-\infty,b)$ *and* $(b-1,\infty)$, *respectively. Then* $T \overset{E}{\sim} \rho(x) = x^p L(x)$ *as* $x \to \infty$ *iff* $S \overset{E}{\sim} \rho(x)$ *as* $x \to \infty$. *Also,* T *is equivalent at infinity with zero iff* S *is equivalent at infinity with zero.*

PROOF. If $T \overset{E}{\sim} \rho(x)$ as $x \to \infty$, then there exist $n > -p$, $a > b$, and a continuous function F on R so that $T = D^n F(x)$ on (a,∞) and (4) holds. By supposition, there exists a polynomial P_k of order $k < n$ so that $B = D^n P_k(x)$ on (b,∞); it is clear that (4) implies that P_k can be chosen so that $k < n+p$. Taking $G(x) := F(x) - P_k(x)$, we obtain $S = D^n G(x)$ on (a,∞), hence $S \overset{E}{\sim} \rho(x)$. Conversely, if $S \overset{E}{\sim} x^p L(x)$ as $x \to \infty$, then there exist $n > -p$, $a > b$ and a continuous function G on R so that $S = D^n G(x)$ and $G(x) \sim C_{p,n} x^{p+n} L(x)$ as $x \to \infty$. Let us observe the iterated convolution $U := H_a^{*m}(-x)*B(x) = \underbrace{H_a(-x)*H_a(-x)* \dots *H_a(-x)}_{m\ times}*B(x)$, which exists in view of the assumption on the support of B. For a sufficiently large $m > n$ U becomes a continuous function $(0,\infty)$, which is zero on (a,∞). Hence $T = D^m G_{m-n}(x)$, on (a,∞), where $G_1(x) := H_a(x)\int_a^x G(t)dt$, $G_k(x) := H_a(x) \int_a^x G_{k-1}(t)dt$ for

$k = 2,3,\ldots,m-n$ and we have $G_{m-n}(x) \sim C_{p,m}x^{p+m}L(x)$ as $x \to \infty$.

We shall omit the proof of the second statement, since it is similar to the proof of the first one.

3. THE QUASIASYMPTOTIC BEHAVIOUR AND THE EQUIVALENCE OF DISTRIBUTIONS AT INFINITY

The quasiasymptotic behaviour of distributions was defined by the Soviet mathematician B.I. Zav'jalov in 1973 and analyzed comprehensively in [3]. It is clear that the quasiasymptotic behaviour and the equivalence at infinity are not comparable in general, since the first is defined only for distributions from S'_+, while the other has no limitations on the support of the observed distribution. Furthermore, a distribution with compact support included in some interval $[0,a]$, $a > 0$, has quasiasymptotic behaviour of order $-m$ for some $m \in \mathbf{N}$, but obviously has no equivalence at infinity in the sense of Definition 2; of course such a distribution tends to zero faster than any degree of x (see Definition 3). Another counterexample is the distribution $T = D^k\delta + H_1(x)x^p$, $k \in \mathbf{N}_0$ and $p < -1$, which acts in the following way:

$$< T;\varphi > : = (-1)^k\varphi^{(k)}(0) + \int_1^\infty \varphi(x)x^p dx, \quad \varphi \in \mathbf{D}.$$

Namely, it has quasiasymptotic behaviour of order -1 related to the function x^{-1}, and by Theorem 1 is equivalent at infinity with the regularly varying function x^p provided that $p \notin Z_-$. It should be emphasized that again the case $p \in Z_-$ is not covered by Definition 2. Let us state two "positive" results.

THEOREM 6. *Let* $T \in S'_+$ *which is not in* E'. *Let us suppose that if* $T=B+S$, *where* $\mathrm{supp}B \subset [0,a]$, $\mathrm{supp}S \subset [a-1,\infty)$, *and* S *has quasiasymptotic behaviour of order* $p > -m$ *related to the regularly varying function* $\rho(x) = x^pL(x)$ ($m \in \mathbf{N}$ *is the quasiasymptotic order of* B). *Then* T *is equivalent at infinity with* $\rho(x)$.

PROOF. It is easy to see that T also has quasiasimptotic behaviour of order p related to $x^p L(x)$; condition $p > -m$ is essential for that. Now, by Theorem I ([3], p. 373) a distribution T has a quasiasymptotic behaviour of order p related to $x^p L(x)$ iff one can find an $n > -p$ such that the iterated convolution $H_0^{*n} * T(x)$ is a continuous function G on R and has an ordinary asymptotic behaviour related to the regularly varying function $x^{p+n} L(x)$ (up to a constant, which is easily seen to be $C_{p,n}$ from (2)). Hence, T is equivalent at infinity with $\rho(x)$.

THEOREM 7. *Let* $T = B + S$, *where these three distributions are as in Theorem 6. If* T *is equivalent at infinity with* $\rho(x) = x^p L(x)$ *for* $p > -1$, *then* T *has quasiasymptotic behaviour of order p related to* $\rho(x)$.

The proof of this statement is straightforward and it is omitted. Let us just observe that the condition $p > -1$ is necessary, as the counterexample at the beginning of the Section shows.

At the end of the paper, let us say that it is possible to give the notion of the asymptotic expansion at infinity which we plan to analyze in a subsequent paper; it is worth noting that a "quasiasymptotic expansion" was defined in [3]. It might be of interest to use the equivalence at infinity,and these expansions in order to obtain statements of the Abelian and Tauberian type for certain generalized integral transformations.

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REZIME

O EKVIVALENCIJI DISTRIBUCIJA U BESKONAČNOSTI

U radu se definiše tzv. "ekvivalencija u beskonačnosti", dokazuju njene osnovne osobine i uporedjuje sa definicijom asimptotskog ponašanja distribucije u beskonačnosti date u [6], kao i kvaziasimptotike date u [3].