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ON σ-PERMUTABLE n-GROUPOIDS

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ABSTRACT

In this paper σ-permutable n-groupoids are defined and considered. An n-groupoid (G,f) is called σ-permutable, where σ is a permutation of the set $\{1,\ldots,n+1\}$, iff $f(x_{\sigma 1},\ldots,x_{\sigma n}) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1,\ldots,x_n) = x_{n+1}$ for all $x_1,\ldots,x_{n+1} \in G$. σ-permutable n-groupoids are a generalization of several classes of n-groupoids. Examples of σ-permutable n-groupoids are given and some of their properties described. Several conditions under which σ-permutable n-groupoids are n-groups are determined.

1. NOTATION AND DEFINITIONS

We shall use the following abreviated notation:

$$f(x_1,\ldots,x_k,x_{k+1},\ldots,x_{k+s},x_{k+s+1},\ldots,x_n) = f(x_1^k,\overset{(s)}{x},x_{k+s+1}^n),$$

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whenever $x_{k+1} = x_{k+2} = \ldots = x_{k+s} = x$ (x_i^j is the empty symbol for $i > j$ and for $i > n$, also $\overset{(0)}{x}$ is the empty symbol).

To avoid repetitions we assume throughout the whole text that $n > 2$.

We say that an n-groupoid (G,f) is k-solvable, where $k \in \{1,\ldots,n\} = N_n$ is fixed, iff the equation

$$f(a_1^{k-1}, x, a_{k+1}^n) = b$$

has a solution $x \in G$ for all elements a_1^n, $b \in G$. If the solution is unique, then (G,f) is called an uniquely k-solvable n-groupoid. If this equation has an unique solution for every $k \in N_n$, then (G,f) is called an n-quasigroup.

An n-groupoid (G,f) is (i,j)-commutative iff

$$f(x_1^{i-1}, x_i, x_{i+1}^{j-1}, x_j, x_{j+1}) = f(x_1^{i+1}, x_j, x_{i+1}^{j-1}, x_i, x_{j+1}^n)$$

for all $x_1^n \in G$ and some fixed $1 \leqslant i < j \leqslant n$. If (G,f) is (i,j)-commutative for every pair (i,j), $i,j \in N_n$, then it is commutative.

An n-groupoid (G,f) is (i,j)-associative, where $1 \leqslant i < j \leqslant n$, iff

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $x_1^{2n-1} \in G$.

An n-groupoid (G,f) is associative (i.e. it is an n-semigroup) iff it is (i,j)-associative for every pair (i,j), $i,j \in N_n$. Note that for the associativity of (G,f) it is sufficient to postulate the $(1,j)$-associativity for all $j \in \{2,3,\ldots,n\} = N_{2,n}$. An associative n-quasigroup is an n-group. An n-semigroup is an n-group iff it is k-solvable for $k = 1$ and $k = n$ or for some k other than 1 and n (see [10], p. 213). In [11] it is proved (but this proof is not complete, cf. [4]) that an n-quasigroup is an n-group iff it is $(i, i+1)$-associative for some $i \in N_{n-1}$.

By S_n we denote the symmetric group of degree n.

If $\sigma \in S_n$, then $x_{\sigma i}, x_{\sigma(i+1)}, \ldots, x_{\sigma j}$ we denote by $x_{\sigma i}^{\sigma j}$. If $i > j$, then $x_{\sigma i}^{\sigma j}$ is considered empty.

An n-quasigroup (Q,f) is called totally symmetric (TS) iff $f(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}$ for all $x_1^{n+1} \in Q$ and all $\sigma \in S_{n+1}$.

2. σ-PERMUTABLE n-GROUPOIDS

DEFINITION 1. *Let* $\sigma \in S_{n+1}$. *An* n-*groupoid* (G,f) *is called* σ-*permutable iff for all* $x_1^{n+1} \in G$

$$f(x_1^n) = x_{n+1} \Longleftrightarrow f(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)}.$$

It is easy to see that this definition can be given in another equivalent form.

DEFINITION 2. *Let* $\sigma \in S_{n+1}$. *If* $\sigma i = n+1$ *for some* $i \in N_n$, *then an* n-*groupoid* (G,f) *is* σ-*permutable iff for all* $x_1^{n+1} \in G$

$$f(x_{\sigma 1}^{\sigma(i-1)}, f(x_1^n), x_{\sigma(i+1)}^{\sigma n}) = x_{\sigma(n+1)}.$$

If $\sigma(n+1) = n+1$, *then* (G,f) *is* σ-*permutable iff for all* $x_1^{n+1} \in G$

$$f(x_{\sigma 1}^{\sigma n}) = f(x_1^n).$$

σ-permutable n-groupoids represent a generalization of various classes of n-groupoids. Among them are i-permutable n-groupoids from [14] and cyclic n-quasigroups from [12]. Indeed, if $\sigma(n+1) = 1$ and $\sigma k = k+1$, for all $k \in N_n$, then a σ-permutable n-groupoid is a cyclic n-quasigroup from [12]. On the other hand, if $\tau k = k$ for $k = 1,\ldots, i-1$, $\tau i = n+1$, $\tau j = j-1$

for $j = i+1,\ldots, n+1$, then a τ-permutable n-groupoid is an i-permutable n-groupoid from [14]. If a permutation $\sigma \in S_{n+1}$ is a cycle $(k,n+1)$, then a σ-permutable n-groupoid is k-invertible in the sense of [3]. Some other classes of n-groupoids which are special cases of σ-permutable n-groupoids will be given later.

As a simple consequence of Definition 1 we obtain:

LEMMA 1. *Let* $\sigma \in S_{n+1}$ *and* H *be a subgroup of* S_{n+1} *generated by* σ*. An* n-*groupoid* (G,f) *is* σ-*permutable iff it is* τ-*permutable for every* $\tau \in H$.

LEMMA 2. *Let* (G,f) *be an* n-*groupoid. The set* H *of all permutations* $\sigma \in S_{n+1}$ *such that*

$$f(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}$$

for all $x_1^{n+1} \in G$, *is a subgroup of* S_{n+1}.

The set H of all such permutations we shall denote by $\Pi(f)$.

The notion of σ-permutable n-groupoids is an extension of the notion of autoparastrophies of n-quasigroups to arbitrary n-groupoids. Namely, if (Q,f) is an n-quasigroup, then a new quasigroup (Q,f^σ) can be defined in the following way:

$$f^\sigma(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}.$$

(Q,f^σ) is said to be a parastrophe (or conjugate) of the n-quasigroup (Q,f). If $f^\sigma = f$, then σ is called an autoparastrophy of (Q,f). Although it is not possible to define a new operation f^σ for an arbitrary n-groupoid as it is done for n-quasigroups, we see that σ-permutable n-groupoids generalize the notion of autoparastrophy to arbitrary n-groupoids.

DEFINITION 3. *Let* H *be a subgroup of* S_{n+1}. *If an* n-*groupoid* (G,f) *is* σ-*permutable for every* $\sigma \in H$, *then* (G,f) *is called a* H-*permutable* n-*groupoid.*

It is obvious that an n-groupoid (G,f) is H-permutable iff it is σ-permutable for all $\sigma \in \Gamma$, where Γ is a set of generators of the group H. From Lemma 1 we see that every σ-permutable n-groupoid is H-permutable, where H is the subgroup of S_{n+1} generated by σ. If (G,f) is H-permutable n-groupoid then it is, of course, K-permutable for every subgroup K of the group H.

A special case of H-permutable n-groupoids are H-n-quasigroups investigated in [9].

Some other examples of H-permutable n-groupoids are the following.

1. Totally symmetric n-quasigroups are H permutable with $H = S_{n+1}$.

2. (i,j)-commutative n-groupoids are H-permutable, where $H = \{(1),(i,j)\}$. Commutative n-groupoids are H-permutable, where $H \simeq S_n$ and $\sigma(n+1) = n+1$ for all $\sigma \in H$.

3. If H is a cyclic subgroup of S_{n+1} generated by the cycle (1,2,...,n), then H-permutable n-groupoid is a cyclic n-quasigroup described in [12].

4. Medial n-groups are H-permutable for a subgroup H of S_{n+1} generated by the cycle (1,n) (see [1]).

5. If a subgroup H of S_{n+1} is generated by the cycle (i,i+1, ..., n+1), then a H-permutable n-groupoid is i-permutable.

6. If H is the alternating subgroup of S_{n+1}, then a H-permutable n-groupoid is an alternating symmetric n-quasigroup from [13].

7. In [8] D.G. Hoffman has given a construction of a H-permutable n-quasigroup (G,f) of order mp, for every $m > n$, $p \geqslant 2$, and every subgroup H of S_{n+1}, such that $\Pi(f) = H$.

PROPOSITION 1. *If* (G,f) *is a* H-*permutable* n-*groupoid, where* H *is a subgroup of* S_{n+1}, *then for every* $\sigma \in H$ *such that* $\sigma(n+1) \neq n+1$ *the* n-*groupoid* (G,f) *is uniquely solvable at the place* $\sigma^{-1}(n+1)$.

PROOF. Let $\sigma \in H$ and let $\sigma i = n+1$. Then by Definition 2 the equation $f(x_{\sigma 1}^{\sigma(i-1)}, y, x_{\sigma(i+1)}^{\sigma n}) = x_{\sigma(n+1)}$ has an uniquely determined solution $y = f(x_1^n)$ for every $x_{\sigma 1}^{\sigma(i-1)}, x_{\sigma(i+1)}^{\sigma(n+1)} \in G$. Hence the H-permutable n-groupoid (G,f) has an unique solution at the place $\sigma^{-1}(n+1)$, which completes the proof.

If σ is a permutation from a subgroup $H \subseteq S_{n+1}$, then $\sigma^{-1} \in H$ and $\sigma^k \in H$ for every $k = 1,2,\ldots$, hence we have the following corollary.

COROLLARY 1. *Every* σ-*permutable* n-*groupoid* (G,f), *where* $\sigma(n+1) \neq n+1$, *is uniquely solvable at the places* $\sigma^k(n+1)$, $k = 1,2,\ldots$, *provided that* $\sigma^k(n+1) \neq n+1$.

COROLLARY 2. *If* H *is a subgroup of* S_{n+1} *such that for every* $j \in N_n$ *there exists* $\sigma \in H$ *such that* $\sigma j = n+1$, *then every* H-*permutable* n-*groupoid is an* n-*quasigroup. In particular, if* H *is a transitive permutation group, then every* H-*permutable* n-*groupoid is an* n-*quasigroup*.

COROLLARY 3. *If* $\sigma(n+1) \neq n+1$, *then every* σ-*permutable commutative* n-*groupoid is a totally symmetric* n-*quasigroup*.

From our Proposition 1 and Proposition 1 from [4] we get the following.

PROPOSITION 2. *A* H-*permutable* n-*groupoid* (G,f) *is an* n-*group if one of the following conditions holds:*

(i) (G,f) *is* (i,i+1)*-associative for some* $2 \leq i \leq n-2$ *and there exist* $\sigma,\tau \in H$ *such that* $\sigma^{-1}(n+1) = i < \tau^{-1}(n+1) \leq n$,

(ii) (G,f) *is* (1,2) *or* (n-1,n)*-associative and there exist* $\sigma,\tau \in H$ *such that* $\sigma(n+1) = 1$ *and* $\tau(n+1) = n$.

PROPOSITION 3. *A* H*-permutable* n*-semigroup* (G,f) *is an* n*-group if one of the following conditions holds:*

(i) *there exists* $\sigma \in H$ *such that* $1 < \sigma(n+1) < n$,

(ii) *there exists* $\sigma,\tau \in H$ *such that* $\sigma(n+1) = 1$ *and* $\tau(n+1) = n$.

(iii) *there exists* $\sigma \in H$ *such that* $\sigma(n+1) = 1$ *and* $\sigma(1) \neq n+1$,

(iv) *there exists* $\sigma \in H$ *such that* $\sigma(n+1) = n$ *and* $\sigma(n) \neq n+1$.

From our Proposition 1, Corollary 1 and Proposition 1 from [4] we obtain:

PROPOSITION 4. *A* σ*-permutable* n*-groupoid* (G,f), *where* $\sigma(n+1) \neq n+1$, *is an* n*-group if one of the following conditions holds:*

(i) *the* (i,i+1)*-associative law holds for* $i = \sigma(n+1)$ *or* $i = \sigma^{-1}(n+1)$, *where* $2 \leq i \leq n-1$, *and* (G,f) *is solvable at some place* $k > i$,

(ii) *the* (i,i+1)*-associative law holds for* $i = \min\{\sigma(n+1),\sigma^{-1}(n+1)\}$, where $2 \leq i \leq n-1$ *and* $\sigma(n+1) \neq \sigma^{-1}(n+1)$,

(iii) $\sigma(n+1) = 1$ *and* $\sigma(n) = n+1$ *(or* $\sigma(n+1) = n$ *and* $\sigma(1) = n+1$) *and* (G,f) *is* (1,2) *or* (n-1,n)*-associative.*

PROPOSITION 5. *A* σ*-permutable* n*-semigroup, where* $\sigma(n+1) \neq n+1$, *is an* n*-group if one of the following conditions holds:*

(i) $\sigma(n+1) = 1$ *and* $\sigma(n) = n+1$,

(ii) $\sigma(n+1) = n$ *and* $\sigma(1) = n+1$,

(iii) $1 < \sigma^{k}(n+1) < n$ *or* $1 < \sigma^{-k}(n+1) < n$ *for some* $k \in \{1,2,3\}$,

(iv) $\sigma(n+1) = 1$ (*or* $\sigma(n+1)=n$) *and* $\sigma^2(n+1) \neq n+1$.

PROPOSITION 6. *Let* (G,f) *be a* σ*-permutable* n*-groupoid and let* $\sigma(n+1) = 1 = \sigma^{-1}(n+1)$. (G,f) *is an* n*-group iff it is* $(1,2)$*-associative and for every* $x \in G$ *there exists* $\hat{x} \in G$ *such that* $f(\overset{(n-i)}{x}, \hat{x}, \overset{(i-2)}{x}, y) = y$ *for all* $y \in G$ *and some fixed* $i \in N_{2,n}$.

PROOF. By a simple generalization of the proof of Theorem 2 from [4] and Corollary 1 from [1], we can prove that a $(1,2)$-associative n-groupoid (G,f) is an n-group iff for all $x \in G$ there exist $\hat{x},\tilde{x} \in G$ such that

$$(1)\qquad f(\overset{(n-i)}{x}, \hat{x}, \overset{(i-2)}{x}, y) = y = f(y, \overset{(j-2)}{x}, \tilde{x}, \overset{(n-j)}{x})$$

for all $y \in G$ and some $i,j \in N_{2,n}$.

Now we have to prove that for a given n-groupoid the first equation from (1) implies the second. Since $\sigma(n+1) = 1$ and $\sigma(1) = n+1$, then by σ-permutability of (G,f) we obtain that the equation $f(y, \overset{(n-1)}{x}) = z$ implies $f(z, \overset{(n-1)}{x}) = y$. Hence $y = f(f(y, \overset{(n-1)}{x}), \overset{(n-1)}{x}) = f(y, f(\overset{(n)}{x}), \overset{(n-2)}{x})$ by the $(1,2)$-associativity. This means that $f(y, \tilde{x}, \overset{(n-2)}{x}) = y$ for all $x,y \in G$ and $\tilde{x} = f(\overset{(n)}{x})$, which completes the proof.

In the same manner we can prove:

PROPOSITION 7. *A* σ*-permutable* n*-groupoid* (G,f), *where* $\sigma(n+1) = n = \sigma^{-1}(n+1)$, *is an* n*-group iff it is* $(n-1,n)$*-associative and for every* $x \in G$ *there exists* $\hat{x} \in G$ *such that* $f(y, \overset{(j-2)}{x}, \hat{x}, \overset{(n-j)}{x}) = y$ *for all* $y \in G$ *and some* $j \in N_{2,n}$.

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REZIME

O σ-PERMUTABILNIM n-GRUPOIDIMA

U radu su definisani i razmatrani σ-permutabilni n-grupoidi. n-grupoid (G,f) se naziva σ-permutabilan, gde je σ premutacija skupa {1,...,n+1}, ako i samo ako je $f(x_{\sigma 1},\ldots, x_{\sigma n}) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1,\ldots,x_n) = x_{n+1}$ za svako $x_1,\ldots,x_{n+1} \in G$. σ-permutabilni n-grupoidi predstavljaju uopštenje više raznih klasa n-grupoida. Navedeni su primeri σ-permutabilnih n-grupoida i opisane neke njihove osobine. Odredjeno je nekoliko uslova pod kojim je σ-permutabilni n-grupoid n-grupa.