ZBORNIK RADOVA
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ON THE LATTICE OF WEAK FUZZY CONGRUENCE RELATIONS ON ALGEBRAS

Gradimir Vojvodić, Branimir Šešelja
Prirodno-matematički fakultet, Institut za matematiku
21000 Novi Sad, dr Ilije Djuričića br. 4, Jugoslavija

ABSTRACT

We consider the set $\overline{C_w(A)}$ of weak L-fuzzy congruence relations on an algebra $A = (A,F)$ (L is a complete lattice), defined for groupoids in [2]. We prove that $(\overline{C_w(A)}, \leq)$ is a complete lattice, having as a sublattice $(\overline{C(A)}, \leq)$, where $\overline{C(A)}$ is a set of fuzzy congruence relations on A ([1]). Moreover, the lattice $(\overline{S(A)}, \leq)$ ($\overline{S(A)}$ is the set of fuzzy subalgebras on A) is a homomorphic image of $(\overline{C_w(A)}, \leq)$.

Let $A = (A,F)$ be an algebra, and let $K \subseteq A$ be the set of its constants (if $K = \emptyset$, then we consider the empty set as a subalgebra of A). Let $L = (L, \wedge, \vee, 0, 1)$ be a complete lattice. All fuzzy sets here are mappings from A (or A^2 in the case of fuzzy relations) to L. The set A and its subsets are identified with their characteristic functions (0 and 1 are from L). Thus, $K : A \to L$, and $K(x) = 1$ if $x \in K$. Otherwise, $K(x) = 0$.

A fuzzy subalgebra of A is any mapping $\bar{B} : A \to L$, such that

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a) $K \subseteq \bar{B}$ [1]), and

b) $\bar{B}(f(x_1,\ldots,x_n)) \geq \bar{B}(x_1) \wedge \ldots \wedge \bar{B}(x_n)$,

for all $x_1,\ldots,x_n \in A$, $f \in F_n \subseteq F$, $n \in N$.

The set of all fuzzy subalgebras on $\mathcal{A}$ is denoted by $\overline{S(\mathcal{A})}$.

<u>A weak fuzzy congruence relation on</u> $\mathcal{A}$ is a mapping $\bar{\rho} : A^2 \to L$, such that ([2]):

(i) For every $c \in K$, $\bar{\rho}(c,c) = 1$ [2]) (reflexivity);

(ii) For all $x,y \in A$, $\bar{\rho}(x,y) = \bar{\rho}(y,x)$ (symmetry);

(iii) For all $x,y \in A$, $\bar{\rho}(x,y) \geq \bigvee_{z \in A} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y))$ (transitivity)

(iv) For all $x_1,\ldots,x_n,y_1,\ldots,y_n \in A$, $f \in F_n \subseteq F$, $n \geq 1$,

$$\bar{\rho}(f(x_1,\ldots,x_n),f(y_1,\ldots,y_n)) \geq \bigwedge_{i=1}^{n} \bar{\rho}(x_i,y_i)$$

(substitution).

The set of all weak fuzzy congruence relations on $\mathcal{A}$ is denoted by $\overline{C_w(\mathcal{A})}$.

If (i) is replaced by

(i′) For every $x \in A$, $\bar{\rho}(x,x) = 1$ (reflexivity),

then $\bar{\rho}$ is a fuzzy congruence relation on $\mathcal{A}$, and the set of all such relations on $\mathcal{A}$ is denoted by $\overline{C(\mathcal{A})}$.

1. PROPOSITION 1.1. $(\overline{C_w(\mathcal{A})}, \leq)$ *is a complete lattice. (The ordering relation is the one defined for fuzzy sets: If* $\bar{\rho}, \bar{\theta} \in \overline{C_w(\mathcal{A})}$, *then* $\bar{\rho} \leq \bar{\theta}$ *iff for all* $x,y \in A$

$$\bar{\rho}(x,y) \leq \bar{\theta}(x,y) \quad .)$$

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1) *This condition can be replaced by the following two:*

a′) $(\forall c \in K)(\forall x \in A)(\bar{B}(c) \geq \bar{B}(x))$, *and*

a″) $(\forall c \in K)(\bar{B}(c) \geq p)$, *where* $p \neq 0$, *and it does not depend on* $\bar{B}$.

2) *If a) in the definition of a fuzzy subalgebra is replaced by a′) and a″), then we require that* $\bar{\rho}(c,c) \geq p$.

P r o o f. The greatest element in the poset $(\overline{C_w(A)}, \leq)$ is A^2, since $A^2(x,y) = 1$, for all x,y.

Let $\{\bar{\rho}_i;\ i \in I\}$ be an arbitrary family from $\overline{C_w(A)}$.

Then

$$\bigcap_{i\in I} \bar{\rho}_i = \bar{\rho}$$

is obviously one weakly reflexive and symmetric fuzzy relation on A. It is also transitive: for every $i \in I$, and $x,y,z \in A$

$$\bar{\rho}_i(x,y) \geq \bar{\rho}_i(x,z) \wedge \bar{\rho}_i(z,y) \geq \bigcap_{i\in I} \bar{\rho}_i(x,z) \wedge \bigcap_{i\in I} \bar{\rho}_i(z,y) ,$$

and thus

$$\bigcap_{i\in I} \bar{\rho}_i(x,y) \geq \bigcap_{i\in I} \bar{\rho}_i(x,z) \wedge \bigcap_{i\in I} \bar{\rho}_i(z,y) ,$$

and for every $z \in A$

$$\bar{\rho}(x,y) \geq \bar{\rho}(x,z) \wedge \bar{\rho}(z,y) .$$

$\bar{\rho}$ satisfies the substitution property: for every $i \in I$, $f \in F_n$, let

$$\bar{\rho}_i(x_1,y_1) = p_i^{(1)}, \ldots, \bar{\rho}_i(x_n,y_n) = p_i^{(n)}.$$

Then obviously

$$\bar{\rho}_i(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) \geq \bigwedge_{k=1}^{n} p_i^{(k)}.$$

For $k = 1,\ldots,n$, let

$$\bar{\rho}(x_k,y_k) = \bigcap_{i\in I} \bar{\rho}_i(x_k,y_k) = \bigwedge_{i\in I} p_i^{(k)} = p^{(k)}. \qquad (*)$$

Then for every $i \in I$

$$\bar{\rho}_i(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) \geq \bigwedge_{k=1}^{n} p_i^{(k)} \geq \bigwedge_{k=1}^{n} \bigwedge_{i\in I} p_i^{(k)} = \bigwedge_{k=1}^{n} p^{(k)}.$$

Hence,

$$\bigwedge_{i\in I} \bar{\rho}_i(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) \geq \bigwedge_{k=1}^{n} p_i^{(k)} ,$$

and by (*)

$$\bar{\rho}(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) \geq \bigwedge_{k=1}^{n} \bar{\rho}(x_k,y_k) .$$

Thus $\overline{C_w(A)}$ is closed under arbitrary intersections, which proves that it is a lattice. □

LEMMA 1.2. *If* $\{\bar{A}_i;\ i\in I\}$ *is a family of fuzzy subalgebras of* A *, then* $\bar{B}=\bigcap_{i\in I}\bar{A}_i$ *is a fuzzy subalgebra of* A*, as well.*

P r o o f. Since for $i\in I$, $f\in F_n$, $x_1,\dots,x_n\in A$,

$$\bar{A}_i(f(x_1,\dots,x_n))\geq \bar{A}_i(x_1)\wedge\dots\wedge\bar{A}_i(x_n)\ ,$$

then

$$\bigwedge_{i\in I}\bar{A}_i(f(x_1,\dots,x_n))\geq \bigwedge_{i\in I}(\bar{A}_i(x_1)\wedge\dots\wedge\bar{A}_i(x_n)) =$$

$$\bigwedge_{i\in I}\bar{A}_i(x_1)\wedge\dots\wedge\bigwedge_{i\in I}\bar{A}_i(x_n)\ .$$

Clearly, $K\subseteq\bar{B}$, completing the proof. □

COROLLARY 1.3. $(\overline{S(A)},\leq)$ *is a complete lattice.*

P r o o f. $\overline{S(A)}$ is closed under arbitrary intersections, and it has the greatest element - the algebra A.

LEMMA 1.4. *If* $\bar{B}\in\overline{S(A)}$*, then* $\bar{B}^2\in\overline{C_w(A)}$.

P r o o f. By definition ([3]), $\bar{B}^2(x,y)=\bar{B}(x)\wedge\bar{B}(y)$. Hence, $\bar{B}^2(x,x)=\bar{B}(x)\geq K(x)$, for every $x\in A$, and thus $\bar{B}^2$ is weakly reflexive. It is obviously symmetric, and we shall prove that it is transitive and satisfies the substitution property:

$$\bar{B}^2(x,z)\wedge\bar{B}^2(z,y)=\bar{B}(x)\wedge\bar{B}(z)\wedge\bar{B}(z)\wedge\bar{B}(y)\ ,$$

and thus

$$\bar{B}^2(x,y)=\bar{B}(x)\wedge\bar{B}(y)\geq\bar{B}^2(x,z)\wedge\bar{B}^2(z,y)\ .$$

To prove the substitution property, take $f\in F_n$ and $x_i,y_i\in A$, $i=1,\dots,n$. Then,

$$\bar{B}^2(f(x_1,\dots,x_n),f(y_1,\dots,y_n)) =$$

$$= \bar{B}(f(x_1,\dots,x_n)) \wedge \bar{B}(f(y_1,\dots,y_n)) \geq$$

$$\geq \bar{B}(x_1)\wedge\dots\wedge\bar{B}(x_n) \wedge \bar{B}(y_1) \wedge \dots \wedge \bar{B}(y_n) =$$

$$= \bigwedge_{i=1}^{n} \bar{B}^2(x_i,y_i) .$$

We omit the proof of the following proposition, since it is similar to the one given (for groupoids) in [2].

PROPOSITION 1.5. *If* $\bar\rho \in \overline{C_w(A)}$, *and if* $\bar{A}_{\bar\rho} : A \to L,\ \ \bar{A}_{\bar\rho}(x) \overset{\mathrm{def}}{=} \bar\rho(x,x)$, *then* $\bar{A}_{\bar\rho} \in \overline{S(A)}$.

Define, now, on the lattice $(\overline{C_w(A)}, \leq)$ a binary reration $\sim$:

$$\bar\rho \sim \bar\theta \ \text{ iff for every } \ x\in A \ \ \bar\rho(x,x) = \bar\theta(x,y) .$$

PROPOSITION 1.6. $\sim$ *is (an ordinary) congruence relation on* $(\overline{C_w(A)}, \leq)$.

P r o o f. $\sim$ is obviously an equivalence relation on $(C_w(A), \leq)$. It satisfies the substitution property: Let

$\bar\rho_1 \sim \bar\theta_1$, that is $\bar\rho_1(x,x) = \bar\theta_1(x,x)$, for every $x \in A$, and

$\bar\rho_2 \sim \bar\theta_2$, that is $\bar\rho_2(x,x) = \bar\rho_2(x,x)$, for every $x \in A$. Then

$$(\bar\rho_1 \wedge \bar\rho_2)(x,x) = (\bar\rho_1 \cap \bar\rho_2)(x,x) = \bar\rho_1(x,x) \wedge \bar\rho_2(x,x) =$$

$$= \bar\theta_1(x,x) \wedge \bar\theta_2(x,x) = (\bar\theta_1 \wedge \bar\theta_2)(x,x) \ ,$$ [1)]

and thus

$$\bar\rho_1 \wedge \bar\rho_2 \sim \bar\theta_1 \wedge \bar\theta_2 .$$

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1) *We use the same symbol "$\wedge$" for the infimum in all the lattices we consider, which can not be confused, since the lattice is here determined by its elements. The same is true with "$\vee$".*

Consider now another binary operations, supremum:

$$\bar{\rho}_1 \vee \bar{\rho}_2 \overset{\text{def}}{=} \cap(\tau;\ \bar{\rho}_1 \cup \bar{\rho}_2 \subseteq \bar{\tau} \in \overline{C_w(A)})\ .$$

Let

$$\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2} : A \to L,\quad \bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}(x) = (\bar{\rho}_1 \cup \bar{\rho}_2)(x,x)\ .$$

Let

$$\bar{B} = \bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}\ ,\ \text{i.e.}\ \ \bar{B} = \cap(\bar{B}_i;\ \bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2} \subseteq \bar{B}_i \in \overline{S(A)}),\ \text{and}$$

$$\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2} : \bar{A} \to L,\quad \bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}(x) = (\bar{\rho}_1 \vee \bar{\rho}_2)(x,x)\ .$$

$\bar{B}$ is a subalgebra generated by the fuzzy set $\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}$, and we shall prove that $\bar{B} = \bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}$ (to prove that $\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}$ is a fuzzy subalgebra as well, see [2]).

First we prove the inclusion $\bar{B} \subseteq \bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}$.

$$\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}(x) = (\bar{\rho}_1 \cup \bar{\rho}_2)(x,x) \leq (\bar{\rho}_1 \vee \bar{\rho}_2)(x,x) = \bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}(x)\ ,$$

and $\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}$ is one of the relations constituing the intersection in $\bar{B}$.

The inverse inclusion, $\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2} \subseteq \bar{B}$ also holds:
By Lemma 1.4., the fuzzy relation $[\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}]^2 = \bar{B}^2$ belongs to $\overline{C_w(A)}$. Then

$$\bar{\rho}_1 \vee \bar{\rho}_2 \subseteq \bar{B}^2,\ \ \text{since}\ \ \bar{\rho}_1 \cup \bar{\rho}_2 \subseteq \bar{B}^2\ .\ \text{Thus}$$

$$\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2}(x) = (\bar{\rho}_1 \vee \bar{\rho}_2)(x,x) \leq \bar{B}^2(x,x) = \bar{B}(x),\ \ \text{that is}$$

$$\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2} \subseteq \bar{B}\ .$$

Let now $\bar{\rho}_1 \sim \bar{\theta}_1$, and $\bar{\rho}_2 \sim \bar{\theta}_2$. Then, for $x \in A$

$$\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}(x) = (\bar{\rho}_1 \cup \bar{\rho}_2)(x,x) = (\bar{\theta}_1 \cup \bar{\theta}_2)(x,x) = \bar{A}_{\bar{\theta}_1 \cup \bar{\theta}_2}(x),\ \ \text{and}$$

thus $\bar{A}_{\bar{\rho}_1 \vee \bar{\rho}_2} = \bar{A}_{\bar{\theta}_1 \vee \bar{\theta}_2}$, since $[\bar{A}_{\bar{\rho}_1 \cup \bar{\rho}_2}] = [\bar{A}_{\bar{\theta}_1 \cup \bar{\theta}_2}]$. Hence,

$\bar\rho_1 \vee \bar\rho_2(x,x) = \bar A_{\bar\rho_1 \vee \bar\rho_2}(x) = \bar A_{\bar\theta_1 \vee \bar\theta_2}(x) = \bar\theta_1 \vee \bar\theta_2(x,x)$, i.e.

$$\bar\rho_1 \vee \bar\rho_2 \sim \bar\theta_1 \vee \bar\theta_2 . \quad \square$$

PROPOSITION 1.7. $(\overline{C(A)}, \leq)$ *is a sublattice of* $(\overline{C_w(A)}, \leq)$ *and* $\overline{C(A)} = [A^2]_\sim$.

P r o o f. Directly by the definition of congruence relation on A, and by the previous proposition. $\square$

PROPOSITION 1.8 $(\overline{C_w(A)}/\sim, \leq) \cong (\overline{S(A)}, \leq)$.

P r o o f. Consider the mapping $h : \overline{C_w(A)}/\sim \to \overline{S(A)}$ such that $h([\bar\rho]_\sim) = \bar A_{\bar\rho}$. h is well defined, i.e. it does not depend on the representative, which can be proved directly. h is onto: Indeed, for every $\bar B \in \overline{S(A)}$, define the fuzzy relation $\bar\rho$ on A, such that

$$\bar\rho(x,y) \overset{def}{=} \begin{cases} \bar B(x), & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

$\bar\rho \in \overline{C_w(A)}$. Indeed $\bar\rho$ is weakly reflexive and symmetric by definition.

It is transitive:

If $x = y$, then for every z, obviously

$$\bar\rho(x,y) \geq \bar\rho(x,z) \wedge \bar\rho(z,y). \tag{1}$$

If $x \neq y$, then $\bar\rho(x,y) = 0$, and (1) holds again.

To prove the substitution property, we also consider two cases:

If for some $i \in \{1,\dots,n\}$, $x_i \neq y_i$, then $\bar\rho(x_i,y_i) = 0$, and thus

$$\bar\rho(f(x_1,\dots,x_n), f(y_1,\dots,y_n)) \geq \bigwedge_{i=1}^{n} \bar\rho(x_i,y_i) = 0.$$

If for every $i \in \{1,\dots,n\}$, $x_i = y_i$, then

$$\bar{\rho}(f(x_1,\ldots,x_n),f(y_1,\ldots,y_n)) = \bar{B}(f(x_1,\ldots,x_n)) \geq$$

$$\bigwedge_{i=1}^{n} \bar{B}(x_i) = \bigwedge_{i=1}^{n} \bar{\rho}(x_i,y_i) .$$

Hence, $\bar{\rho} \in \overline{C_w(A)}$, and $h([\rho]_\sim) = \bar{A}_{\bar{\rho}} = \bar{B}$ (since $\bar{A}_{\bar{\rho}}(x) = \bar{\rho}(x,x) =$ $= \bar{B}(x))$.

h is "1 - 1": If $[\bar{\rho}]_\sim \neq [\bar{\theta}]_\sim$, then $\bar{\rho}(x,x) \neq \bar{\theta}(x,x)$ for at least one $x \in A$.

If $h([\bar{\rho}]_\sim) = \bar{A}_{\bar{\rho}}$ and $h([\bar{\theta}]_\sim) = \bar{A}_{\bar{\rho}}$, then

$$\bar{A}_{\bar{\rho}}(x) = \bar{\rho}(x,x) \neq \bar{\theta}(x,x) = \bar{A}_{\bar{\rho}}(x,x) , \text{ i.e. } \bar{A}_{\bar{\rho}} \neq \bar{A}_{\bar{\theta}} .$$

h preserves the ordering: If $[\bar{\rho}]_\sim \subseteq [\bar{\theta}]_\sim$ then $\bar{\rho} \cap \bar{\theta} \sim \bar{\rho}$, and thus $\bar{\rho} \cap \bar{\theta}(x,x) = \bar{\rho}(x,x)$, i.e.

$$\bar{\rho}(x,x) \wedge \bar{\theta}(x,x) = \bar{\rho}(x,x) ,$$

which means that $\bar{A}_{\bar{\rho}}(x) \wedge \bar{A}_{\bar{\theta}}(x) = \bar{A}_{\bar{\rho}}(x)$, and hence $\bar{A}_{\bar{\rho}} \subseteq \bar{A}_{\bar{\theta}}$.

h^{-1} preserves the ordering: Let $\bar{B} \subseteq \bar{D}$, and $h^{-1}(\bar{D}) = [\bar{\rho}]_\sim$. Then $\bar{\rho}(x,x) = \bar{B}(x)$, and $\bar{\theta}(x,x) = \bar{D}(x)$. Hence,

$$\bar{\rho}(x,x) \wedge \bar{\theta}(x,x) = \bar{\rho}(x,x) , \text{ i.e.}$$

$$\bar{\rho} \cap \bar{\theta}(x,x) = \bar{\rho}(x,x), \text{ and thus } \bar{\rho} \cap \bar{\theta} \sim \bar{\theta} .$$

Finally we have $[\rho \cap \theta]_\sim = [\bar{\rho}]_\sim \cap [\bar{\theta}]_\sim = [\bar{\rho}]_\sim$, and

$$[\bar{\rho}]_\sim \subseteq [\bar{\theta}]_\sim . \quad \square$$

REFERENCES

[1] Vojvodić, G., Šešelja B., *O strukturi slabih relacija ekvivalencije i slabih relacija kongruencije*, *Mat. Vesnik, 1(14)(29), Beograd, 1977, 147-152.*

[2] Šešelja, B., Vojvodić, G., *Fuzzy Congruence Relations and Groupoids, Univ. u Novom Sadu, Zb. rad. Prir.-mat. fak., Ser. mat., 15, 1(1985), 209-215.*

[3] Delorme,M., *Sous-groupes flous, Seminaire."Mathematique floue", Lion, 1978-79.*

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REZIME

O MREŽI SLABIH RASPLINUTIH KONGRUENCIJA NA ALGEBRAMA

Razmatra se skup slabih L-rasplinutih kongruencija date algebre, gde je $\mathcal{L}$ kompletna mreža. Pokazuje se da je taj skup, u odnosu na uobičajeni poredak rasplinutih skupova, kompletna mreža. Jednu njenu podmrežu obrazuju sve rasplinute kongruencije date algebre. Pored toga, mreža rasplinutih podalgebri iste algebre je homomorfna slika mreže slabih kongruencija (u radu navedenom pod [1], termin "slab" korišćen je u smislu "rasplinut").