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ON THE PERMUTABILITY OF WEAK FUZZY CONGRUENCE RELATIONS

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ABSTRACT

We consider the composition "o" of fuzzy binary relations on the set $\overline{C_w(A)}$ of weak fuzzy congruence relations on the given algebra A, using the complete lattice L. (Weak fuzzy congruence relations are defined for groupoids in [1], and it was proved in [2] that $(\overline{C_w(A)},\leq)$ is a complete lattice, having as a homomorphic image the lattice of all fuzzy subalgebras of A).

We prove that, provided that L is infinitely distributive, $(\overline{C_w(A)},o)$ is a semilattice iff all weak fuzzy congruence relations on A are permutable. Permutability, on the other hand, does not imply the equality $\bar\rho o \bar\theta = \bar\rho \vee \bar\theta$ in the lattice $(\overline{C_w(A)},\leq)$. Here we give the necessary and sufficent conditions for that equality, and we describe the connection between the operations o and V in all other cases.

1. Let $A = (A,F)$ be an algebra, $K \subseteq A$ the set of its constants, and $L = (L,\wedge,\vee,0,1)$ a complete lattice.

a) Every mapping $\bar\rho : A^2 \to L$ is a fuzzy relation on A.

b) A fuzzy relation $\bar\rho$ on A is a weak fuzzy congruence relation on A, if the following is satisfied:

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If $c \in K$, then $\bar{\rho}(c,c) = 1$ (weak reflexivity);

For $x,y \in A$, $\bar{\rho}(x,y) = \bar{\rho}(y,x)$ (symmetry);

For $x,y \in A$, $\bar{\rho}(x,y) \geq \bigvee_{z \in A} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y))$ (transitivity);

For $x_1,\dots,x_n,y_1,\dots,y_n \in A$, $f \in F_n \subseteq F$,

$\bar{\rho}(f(x_1,\dots,x_n), f(y_1,\dots,y_n)) \geq \bigwedge_{i=1}^{n} \bar{\rho}(x_i,y_i)$ (substitution).

If $\overline{C_w(A)}$ is a set of all weak fuzzy congruence relations on A, then:

1.1. $(\overline{C_w(A)}, \leq)$ is a complete lattice (where $\bar{\rho} \leq \bar{\theta}$ iff for every $x,y \in A$, $\bar{\rho}(x,y) \leq \bar{\theta}(x,y)$). ([2]).

c) If $\bar{\rho}$ and $\bar{\theta}$ are two arbitrary fuzzy relations on A, then $\bar{\rho} \circ \bar{\theta} : A^2 \to L$, and for $x,y \in A$,

$$\bar{\rho} \circ \bar{\theta}(x,y) = \bigvee_{z} (\bar{\rho}(x,z) \wedge \bar{\theta}(z,y)) .$$

1.2. [3] If L is complete and infinitely distributive, then $\circ$ is an associative operation in the set of all fuzzy relations on A.

In the following, L is a complete lattice, and it is distributive if explicitly stated.

2. Here we consider the conditions for $\circ$ to be an operation on $\overline{C_w(A)}$.

PROPOSITION 2.1. *Let L be an infinitely distributive lattice. Then for $\bar{\rho},\bar{\theta} \in \overline{C_w(A)}$: $\bar{\rho} \circ \bar{\theta} \in \overline{C_w(A)}$* iff $\bar{\rho} \circ \bar{\theta} = \bar{\theta} \circ \bar{\rho}$.

P r o o f. Let $\bar{\rho} \circ \bar{\theta} = \bar{\theta} \circ \bar{\rho}$. Then, for $c \in K$

$$\bar{\rho} \circ \bar{\theta}(c,c) = \bigvee_{z \in A} (\bar{\rho}(c,z) \wedge \bar{\theta}(z,c)) = \bar{\rho}(c,c) \wedge \bar{\theta}(c,c) = 1$$

(reflexivity)

$$\bar{\rho} \circ \bar{\theta}(x,y) = \bigvee_{z} (\bar{\rho}(x,z) \wedge \bar{\theta}(z,y)) = \bigvee_{z} (\bar{\theta}(y,z) \wedge \bar{\rho}(z,x)) =$$

$$= \bar{\theta} \circ \bar{\rho}(y,x) = \bar{\rho} \circ \bar{\theta}(y,x) \quad \text{(symmetry)}.$$

Composition o is associative, since L is infinitely distributive, and thus $(\bar\rho \circ \bar\theta)\circ(\bar\rho \circ \bar\theta) = (\bar\rho \circ \bar\rho)\circ(\bar\theta \circ \bar\theta) \leq \bar\rho \circ \bar\theta$. We use here the fact that $\bar\rho_1 \subseteq \bar\rho$ and $\bar\theta_1 \subseteq \bar\theta$ imply $\bar\rho_1 \circ \bar\theta_1 \subseteq \bar\rho \circ \bar\theta$ (by the definition of o). (transitivity).

Let $f \in F$ be a binary operation denoted by ".". Then, by the distributivity of L,

$$\bar\rho \circ \bar\theta(x,y) \wedge \bar\rho \circ \bar\theta(u,v) = (\bigvee_z(\bar\rho(x,z) \wedge \bar\theta(z,y)) \wedge (\bigvee_t(\bar\rho(u,t)\wedge\bar\theta(t,v))$$

$$= \bigvee_z \bigvee_t(\bar\rho(x,z) \wedge \bar\theta(z,y) \wedge \bar\rho(u,t) \wedge \bar\theta(t,v)) \leq$$

$$\leq \bigvee_{z,t} (\bar\rho(x\cdot u,z\cdot t) \wedge \bar\theta(z\cdot t,y\cdot v)) \leq \bigvee_p(\bar\rho(x\cdot u,p) \wedge \bar\theta(p,y\cdot v) =$$

$$= \bar\rho \circ \bar\theta(x\cdot u,y\cdot v) .$$

Hence, $\bar\rho \circ \bar\theta \in \overline{C_w(A)}$.

Let now $\bar\rho \circ \bar\theta \in \overline{C_w(A)}$. Then

$$\bar\rho \circ \bar\theta(x,y) = \bar\rho \circ \bar\theta(y,x) = \bigvee_z(\bar\rho(y,z) \wedge \bar\theta(z,x)) =$$

$$= \bigvee_z(\bar\theta(x,z) \wedge \bar\rho(z,y)) = \bar\theta \circ \bar\rho(x,y), \text{ i.e. } \bar\rho \circ \bar\theta = \bar\theta \circ \bar\rho . \square$$

PROPOSITION 2.2. *Let* $\bar\rho, \bar\theta$ *as well as* $\bar\rho \circ \bar\theta \in \overline{C_w(A)}$. *Then* $\bar\rho \vee \bar\theta = \bar\rho \circ \bar\theta$ *iff for all* $x \in A, \bar\rho(x,x) = \bar\theta(x,x)$.

P r o o f. Let for $x \in A$ $\bar\rho(x,x) = \bar\theta(x,x)$. Then

$$\bar\rho \cup \bar\theta \subseteq \bar\rho \circ \bar\theta \subseteq \bar\rho \vee \bar\theta .$$

Indeed

$$\bar\rho \cup \bar\theta(x,y) = \bar\rho(x,y) \vee \bar\theta(x,y) = (\bar\rho(x,y) \wedge \bar\rho(y,y)) \vee$$
$$\vee (\bar\theta(x,x) \wedge \bar\theta(x,y)) = (\bar\rho(x,y) \wedge \bar\theta(y,y)) \vee (\bar\rho(x,x) \wedge \bar\theta(x,y)) \leq$$

$$\leq \bigvee_z(\bar\rho(x,z) \wedge \bar\theta(z,y)) = \bar\rho \circ \bar\theta(x,y), \text{ since for } t,u \in A$$

$$\bar\rho(t,t) = \bar\theta(t,t) \geq \bar\theta(t,u) .$$

Hence $\bar\rho \cup \bar\theta \subseteq \bar\rho \circ \bar\theta$. We have also

$\bar\rho(x,z) < \bar\rho \cup \bar\theta(x,z) < \bar\tau(x,z)$, for every $\bar\tau \in \overline{C_w(A)}$, satisfying $\bar\rho \cup \bar\theta \subseteq \bar\tau$. Similary, $\bar\theta(z,y) < \bar\rho \cup \bar\theta(z,y) < \bar\tau(z,y)$, for the same $\bar\tau$. Thus,

$$\bar\rho \circ \bar\theta(x,y) = \bigvee_z (\bar\rho(x,z) \wedge \bar\theta(z,y)) < \bigvee_z (\bar\tau(x,z) \wedge \bar\tau(z,y)) < \bar\tau(x,y),$$

since $\bar\tau$ is transitive. Hence

$$\bar\rho \circ \bar\theta \subseteq \bigcap(\bar\tau;\ \bar\rho \cup \bar\theta \subseteq \bar\tau) = \bar\rho \vee \bar\theta.$$

Since $\bar\rho \circ \bar\theta \in \overline{C_w(A)}$, $\bar\rho \circ \bar\theta \subset \bar\rho \vee \bar\theta$ is impossible, and thus

$$\bar\rho \circ \bar\theta = \bar\rho \vee \bar\theta.$$

Let now $\bar\rho \vee \bar\theta = \bar\rho \circ \bar\theta$, which means that

$$\bar\rho \circ \bar\theta > \bar\rho \quad \text{and} \quad \bar\rho \circ \bar\theta > \bar\theta .$$

It is also true that

$$\bar\rho \circ \bar\theta(x,x) = \bigvee_z (\bar\rho(x,z) \wedge \bar\theta(z,x)) = \bar\rho(x,x) \wedge \bar\theta(x,x)$$

(since $\bar\rho(t,t) > \bar\rho(t,u)$). Thereby

$$\bar\rho \circ \bar\theta(x,x) = \bar\rho(x,x) = \bar\theta(x,x). \quad \square$$

3. In this part we define for every subset T of $\overline{C_w(A)}$ a special mapping $f_T : T \to \overline{C_w(A)}$, such that

$$\bigvee(\bar\rho;\ \bar\rho \in T) = \bigvee(f_T(\bar\rho);\ \bar\rho \in T).$$

We use f_T to describe the connection between $\circ$ and $\vee$ in $\overline{(C_w(A), <)}$.

Let $T = \{\bar\rho_i;\ i \in I\} \subseteq \overline{C_w(A)}$, $I \neq \emptyset$. For $\bar\rho \in \overline{C_w(A)}$, let

$$\bar\delta_{\bar\rho} : A^2 \to L,\quad \bar\delta_{\bar\rho}(x,y) = \begin{cases} \bar\rho(x,y), & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Let also $\bar\Delta_T \overset{\text{def}}{=} \bigvee_{i \in I} \bar\delta_{\bar\rho_i}$, and for $\bar\rho \in T$

(*) $$f_T(\bar\rho) \overset{\text{def}}{=} \bigcap(\bar\tau;\ \bar\tau \in \overline{C_w(A)},\ \bar\Delta_T \cup \bar\rho \subseteq \bar\tau).$$

LEMMA 3.1. *If* $\bar{\rho} \in T$, *then* $\bar{\rho} \le f_T(\bar{\rho})$.

P r o o f. $\bar{\rho} \subseteq \bar{\Delta}_T \cup \bar{\rho} \subseteq \bar{\tau}$, for all $\bar{\tau}$ constituting the intersection in (*). □

LEMMA 3.2. *If* $\bar{\rho} \in T$, *then* $f_T(\bar{\rho}) \le \vee(\bar{\rho}_i;\ \bar{\rho}_i \in T)$.

P r o o f. $\bar{\Delta}_T \cup \bar{\rho} \le \bigvee_{i\in I} \bar{\rho}_i$, and this supremum is one of the relations constituting the above intersection. Thus

$$f_T(\bar{\rho}) \le \bigvee_{i\in I} \bar{\rho}_i \ . \quad \square$$

LEMMA 3.3. *a) If* $\bar{\rho} \in T$, *then* $\bar{\delta}_{f_T(\bar{\rho})} = \bar{\Delta}_T$

b) If $\bar{\rho}, \bar{\theta} \in T$, *then* $\bar{\delta}_{f_T(\bar{\rho})} = \bar{\delta}_{f_T(\bar{\theta})}$.

P r o o f. a) Let $\bar{A}^2_{\bar{\Delta}_T} : A^2 \to L$, such that

$$\bar{A}^2_{\bar{\Delta}_T}(x,y) = \bar{\Delta}_T(x,x) \wedge \bar{\Delta}_T(y,y) \ .$$

Then $\bar{A}^2_{\bar{\Delta}_T} \in \overline{C_w(A)}$ ([2]), and $\bar{A}^2_{\bar{\Delta}_T}(x,x) \ge \bar{\delta}_{\bar{\rho}}(x,y)$.

Thus, $\bar{A}^2_{\bar{\Delta}_T}$ is one of the relations constituting the intersection in (*), and hence $f_T(\bar{\rho}) \le \bar{A}^2_{\bar{\Delta}_T}$. That implies

$$\bar{\delta}_{f_T(\bar{\rho})}(x,x) = f_T(\bar{\rho})(x,x) \le \bar{A}^2_{\bar{\Delta}_T}(x,x) = \bar{\Delta}_T(x,x), \text{ i.e.}$$

(1) $$\bar{\delta}_{f_T(\bar{\rho})} \le \bar{\Delta}_T \ .$$

Conversely, (*) implies $\bar{\Delta}_T \le \bar{\tau}$, for every $\bar{\tau}$ from the family constituting the intersection, and

$$\bar{\Delta}_T \subseteq \cap\, \bar{\tau} = f_T(\bar{\rho}) \ .$$

Thus, $\bar{\Delta}_T(x,x) \le f_T(\bar{\rho})(x,x) = \bar{\delta}_{f_T(\bar{\rho})}(x,x)$, i.e.

(2) $$\bar{\Delta}_T \le \bar{\delta}_{f_T(\bar{\rho})} \ .$$

By (1) and (2)

$$\bar{\delta}_{f_T(\bar{\rho})} = \bar{\Delta}_T .$$

b) Directly follows by a). □

PROPOSITION 3.4. *If* $T = \{\bar{\rho}_i ;\ i \in I\} \subseteq \overline{C_w(A)}$, *then*

$$\bigvee_{i \in I} \bar{\rho}_i = \bigvee_{i \in I} f_T(\bar{\rho}_i) .$$

P r o o f. By Lemma 3.2., for every $\bar{\rho} \subseteq T$, $f_T(\bar{\rho}) <$ $< \bigvee_{i \in I} \bar{\rho}_i$, and

$$\bigvee_{i \in I} f_T(\bar{\rho}_i) < \bigvee_{i \in I} \bar{\rho}_i .$$

By Lemma 3.1., for every $i \in I$, $\bar{\rho}_i < f_T(\bar{\rho}_i)$, and

$$\bigvee_{i \in I} \bar{\rho}_i < \bigvee_{i \in I} f_T(\bar{\rho}_i) .$$

Thus, $\bigvee_{i \in I} \bar{\rho}_i = \bigvee_{i \in I} f_T(\bar{\rho}_i)$. □

PROPOSITION 3.5. *If L is an infinitely distributive lattice, and the set* $T = \{\bar{\rho}, \bar{\theta}\} \subseteq \overline{C_w(A)}$ *satisfies the equality*

$$f_T(\bar{\rho}) \circ f_T(\bar{\theta}) = f_T(\bar{\theta}) \circ f_T(\bar{\rho}) ,$$

then

$$\bar{\rho} \vee \bar{\theta} = f_T(\bar{\rho}) \circ f_T(\bar{\theta}) .$$

P r o o f. By Proposition 2.1., $f_T(\bar{\rho}) \circ f_T(\bar{\theta}) \in$ $\in \overline{C_w(A)}$, by Lemma 3.3. b) $\bar{\delta}_{f_T(\bar{\rho})} = \bar{\delta}_{f_T(\bar{\theta})}$, by Proposition 2.2.

$$f_T(\bar{\rho}) \vee f_T(\bar{\theta}) = f_T(\bar{\rho}) \circ f_T(\bar{\theta})$$

, and by Proposition 3.4. (a) is satisfied, proving the proposition. □

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REZIME

O PERMUTABILNOSTI SLABIH RELACIJA

KONGRUENCIJE

U radu se ispituje kompozicija rasplinutih relacija na mreži slabih rasplinutih kongruencija date algebre. Daju se potrebni i dovoljni uslovi da to bude asocijativna operacija, kao i potrebni i dovoljni uslovi da kompozicija dve slabe rasplinute kongruencije bude supremum u odgovarajućoj mreži.