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SYMMETRIC FUNCTIONS IN A MAXIMAL SET OF THREE-VALUED LOGICAL FUNCTIONS

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ABSTRACT

Miyakawa has proved that the maximal set $\mathrm{Pol}\begin{pmatrix}0&1&2&1&0&1&2\\0&1&2&0&1&2&1\end{pmatrix}$ of functions of P_3 is divided into 54 classes of functions. In this paper we show that each of these classes contains a symmetric function. Thus classes of bases and S-bases of the set coincide .

1. INTRODUCTION

The set of three-valued logical functions, i.e. the union of all function $\{f \mid E_3^n \to E_3\}$ for $E_3 = \{0,1,2\}$ and $n = 0,1,2, \ldots$ is denoted by P_3.

A subset F of P_3 is said to be closed if it does not yield a function which is not in F by means of superposition (e.g. see [1]) among functions in F.

For closed sets F and H such that $F \subset H$ (proper inclusion), F is H-maximal if there is no closed set G such that $F \subset G \subset H$.

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A subset of functions in H is complete in H if every function in H can be represented as a superposition of the elements of the set. If the number of H-maximal sets is finite then a subset of functions is complete in H if it is not contained in any H-maximal set ([1]). A finite complete set of functions in H is called a base in H, if none of it's subsets is complete in H.

A set of functions $\{f_1,\dots,f_s\}$ is called pivotal set in H, iff for every function f_i $(1 \leq i \leq s)$ there exists a H-maximal set which does not contain the function f_i, and all the other functions $f_1,\dots,f_s$ are elements of this H-maximal set. From this definitions it follows that the base is a complete pivotal set of functions.

The rank of the base (pivotal set) is the number of elements of the base (pivotal set).

Let m be the number of H-maximal sets. A function f is of the class $a_1,\dots,a_m$, $a_i \in \{0,1\}$, $1 \leq i \leq m$, where $a_i = 0$ iff the function f is an element of the i-th H-maximal set $(1 \leq i \leq m)$. Classes of function for each base determine the class of the given base. Analogously the classes of pivotal sets are defined.

We shall recall some notations of functions preserving an h-ary relation $\underline{X}$. We denote it by a matrix, i.e. $\underline{X}^t \subset E_3^h$. Then for h-ary vectors $a ,\dots,a_h$ $(a_i \in E_3^n)$,

$$\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \in \underline{X} \iff \text{for all } i, \begin{pmatrix} a_{1i} \\ \cdot \\ \cdot \\ \cdot \\ a_{ni} \end{pmatrix} \in \underline{X}.$$

Then the set of functions preserving $\underline{X}$ (denoted by Pol $\underline{X}$) is defined by

$$X = \text{Pol } \underline{X} = \left\{ f \,\middle|\, \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_h \end{pmatrix} \in X \Rightarrow \begin{pmatrix} f(a_1) \\ \cdot \\ \cdot \\ \cdot \\ f(a_h) \end{pmatrix} \in \underline{X} \right\}.$$

The intersection $X_1 \cap \ldots \cap X_k$ will be denoted by $X_1 \ldots X_k$.

The set P_3 has exactly 18 maximal sets ([1]). Maximal sets of these maximal sets are investigated in [2].

THEOREM 1 ([2]). *B_1 has exactly the following 7 maximal sets* ($B_1 = \mathrm{Pol}\binom{0121012}{0120121}$):

$$I_1 = B_1\mathrm{Pol}(1) \qquad I_{01} = B_1\mathrm{Pol}(0\ 1)$$

$$I_{12} = B_1\mathrm{Pol}(0\ 1) \qquad I_{02} = B_1\mathrm{Pol}(0\ 2)$$

$$B_5 = \mathrm{Pol}\begin{pmatrix}0 & 1 & 2 & 1 & 1\\ 0 & 1 & 2 & 0 & 2\end{pmatrix} \qquad B_6 = \mathrm{Pol}\begin{pmatrix}0 & 1 & 2 & 1 & 0 & 2 & 1 & 2\\ 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0\end{pmatrix}$$

$$B_7 = \mathrm{Pol}\begin{pmatrix}1 & 2 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\ 2 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0\end{pmatrix}.$$

Clases of functions of B_1 are determined in [3].

THEOREM 2 ([3]). *The number of nonempty classes of functions of the set B_1 is 54.*

These classes are presented in Table 1. The components in the characteristic vectors and the B_1-maximal sets in Theorem 1 are in the same order.

Interchaning 0 and 2 in the definition of all the B_1-maximal sets I_1, I_{01}, I_{12}, I_{02}, B_5, B_6 and B_7 they are mapped on the sets I_1, I_{12}, I_{01}, I_{02}, B_5, B_6 and B_7 respectively. The set B_1 is mapped into B_1. Classes of functions c_1 and c_2 are similar if c_1 can be obtained from c_2 by above transformation of B_1-maximal sets. The number of nonsimilar classes of B_1 is 39. For the remaining 15 classes the corresponding similar classes are given in the Table 1.

1. 0111111	2. 0110111	3. 0101111
4. 0011111 s-3	5. 0111011	6. 0111110
7. 0100111	8. 0010111 s-7	9. 0001111
10. 0110011	11. 0101011 s-12	12. 0011011
13. 0110110	14. 0101110 s-15	15. 0011011
16. 0111010	17. 1101100 s-18	18. 1011100
19. 0000111	20. 0100011 s-21	21. 0010011
22. 0001011	23. 0001101	24. 0100110 s-25
25. 0010110	26. 0001110	27. 0110010
28. 0101010 s-29	29. 0011010	30. 1100100 s-31
31. 1010100	32. 0101100 s-33	33. 0011100
34. 0000011	35. 0000101	36. 0001001
37. 0000110	38. 0100010 s-39	39. 0010010
40. 0001010	41. 0100100 s-42	42. 0010100
43. 0001100	44. 1100000 s-45	45. 1010000
46. 0101000 s-47	47. 0011000	48. 0000001
49. 0000010	50. 0000100	51. 0100000 s-52
52. 0010000	53. 0001000	54. 0000000

Table 1

2. SYMMETRIC FUNCTIONS

An n-ary function $f(x_1, \ldots, x_n)$ is said to be symmetric iff the following equality is valid:

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n),$$

where $(y_1, \ldots, y_n)$ is an arbitrary permutation of $(x_1, \ldots, x_n)$.

Symmetric functions have algebraic properties which make it desirable to treat them as a separate class.

s-base (s-pivotal set) is a base (pivotal set) which contains only symmetric functions.

It follows from the definition that the value of n-ary symmetric functions for vectors which contain the same number of 0, same number of 1 and same number of 2 is equal. Hence, we define

$$f[m_0, m_1, m_2] = f(0^{m_0}, 1^{m_1}, 2^{m_2}), \; m_0 + m_1 + m_2 = n.$$

The number of n-ary symmetric functions of P_3 is $3^{\binom{n+2}{2}}$ ([4]).

The number of n-ary symmetric functions of the set B_1 was investigated in [4]. It is proved in [4] that the set B_1 contains

$$s_n(B_1) = 1 + 2 \sum_{k=1}^{n+1} h(k)$$

n-ary functions, where

$$h(0) = 1,$$

$$h(n) = \sum_{k=1}^{n} a_k h(n-k) \quad \text{for} \quad n > 0$$

$$a_k = 2^{\binom{k+1}{2}} - 2^{\binom{k}{2}}.$$

For $n \leq 4$ we obtain the following data:

n	0	1	2	3	4
$s_n(B\ )$	3	17	155	2409	72721

We show in this paper that each of 54 classes of functions of B_1 contains a symmetric function.

3. SYMMETRIC FUNCTION REPRESENTATIVES FOR THE CLASSES

The constant 0 is an example of the class 45 and the constant 1 is an example of the class 53.

Let S_{abc} be the unary function for which the following is satisfied:

$$S_{abc}(0) = a,\ S_{abc}(1) = b \text{ and } S_{abc}(2) = c.$$

Unary functions are examples of some classes:

S_{001}(class 18), S_{210}(class 27), S_{110}(class 47),
S_{010}(class 52), S_{012}(class 54).

Examples of binary symmetric functions of some classes are presented in the following table.

class	f(00)	f(11)	f(22)	f(01)	f(02)	f(12)
1	1	1	0	2	1	1
10	2	1	0	1	0	0
16	2	1	0	1	1	1
29	1	1	0	1	2	1
33	1	1	1	1	1	0
39	0	1	0	1	2	1
43	1	1	1	0	1	1
48	0	1	2	0	0	1

Table 2.

The table 3 gives some examples of ternary symmetric functions of some classes.

class	f(000)	f(111)	f(222)	f(001)	f(011)	f(002)	f(002)	f(112)	f(122)	f(012)
6	1	1	0	2	1	1	1	1	1	1
23	0	1	2	1	0	1	1	1	1	1
31	0	0	0	1	1	0	0	0	1	1
35	0	1	2	1	0	0	0	1	1	1
40	1	1	2	1	1	2	0	1	1	1
42	0	1	0	1	1	0	0	0	1	1
49	0	1	2	1	1	2	0	1	1	1

Table 3.

Examples of symmetric functions with 4 variables are given in the following table.

class														
5	2	1	0	1	1	1	1	2	0	1	1	0	1	1	1
12	1	1	0	1	1	1	0	2	0	1	1	0	1	1	1
15	1	1	1	1	1	1	2	1	1	1	1	0	1	1	1
21	0	1	0	1	1	1	0	2	0	1	1	0	1	1	1
22	1	1	2	1	1	1	2	0	2	1	1	2	1	1	1
34	0	1	2	1	1	1	2	0	2	1	1	2	1	1	1
36	0	1	2	0	1	1	0	2	1	1	1	1	1	1	1

Table 4.

In the above table the columns denote the values f(0000), f(1111), f(2222), f(0001), f(0011), f(0111), f(0022), f(0222), f(1112), f(1122), f(1222), f(0012), f(0112) and f(0122) respectively.

Examples of symmetric functions with 5, 6 and 11 variables of some classes are the following:

class 13:

f(0,0,0,0,0) = 2,f(2,2,2,2,2) = f(1,1,2,2,2) = = f(0,0,0,0,2) = f(0,0,0,2,2) = f(0,0,2,2,2) = = f(0,2,2,2,2) = 0, f(x) = 1 otherwise.

class 50:

f(0,0,0,0,0) = 0,f(2,2,2,2,2) = f(1,1,2,2,2) = = f(0,0,0,0,2) = f(0,0,0,2,2) = f(0,0,2,2,2) = = f(0,2,2,2,2) = 2, f(x) = 1 otherwise.

class 2:

f(2,2,2,2,2,2,2) = f(0,2,2,2,2,2) = f(0,0,0,2,2,2) = = f(1,2,2,2,2,2) = 0, f(0,0,0,0,0,0) = f(0,0,0,0,2, 2) = f(0,0,0,0,0,2) = f(1,1,0,0,0,0) = f(0,0,2,2,2, 2) = 2, f(x) = 1 otherwise.

class 3:

f(0,0,2,2,2,2) = 0, f(1,2,2,2,2,2) = f(0,0,0,2,2,2) = = f(1,1,0,0,0,0) = f(0,0,0,0,0,2) = f(0,0,0,0,2,2) = = f(0,2,2,2,2,2) = f(2,2,2,2,2,2) = 2, f(x) = 1 otherwise.

class 7:

f(0,0,0,2,2,2) = 0, f(0,0,0,0,0,0) = f(0,0,0,0,0,2) = = f(0,0,0,0,2,2) = f(0,0,2,2,2,2) = f(0,2,2,2,2,2) = = f(2,2,2,2,2,2) = f(1,1,0,0,0,0) = 2, f(x) = 1 otherwise.

class 9:

f(0,0,2,2,2,2) = f(1,1,0,0,0,0) = f(0,0,0,2,2,2) = 2, f(x) = 1 otherwise.

class 19:

f(0,0,2,2,2,2) = f(0,0,0,0,0,2) = f(0,0,0,0,2,2) = = f(0,0,0,0,0,0) = 1,1,0,0,0,0) = 0, f(0,0,0,2,2,2) = = f(0,2,2,2,2,2) = f(2,2,2,2,2,2) = 2, f(x) = 1 otherwise.

class 25:

f[5,0,6] = 2, f[0,2,9] = 0, f[k,0,11-k] = 0 for $0 \leq k \leq 11$ and $k \neq 5$, f(x) = 1 otherwise.

class 26:
f[0,2,9] = f[0,0,11] = f[6,0,5] = 2, f[11,0,0] = = f[5,0,6] = 0, f(x) = 1 otherwise.
class 37:
f[11,0,0] = f[5,0,6] = 0, f[0,2,9] = 2, f[k,0,11-k] = = 2 for $0 \leq k \leq 10$ and $k \neq 5$, f(x) = 1 otherwise.

Miyakawa also determined (by using the computer) the number of classes of bases and pivotal incomplete sets of each rank of the set B_1. The datas obtained in [3] are presented in the Table 5.

rank	1	2	3	4	5	6	Σ
numbers of classes of bases	0	28	999	2831	724	17	4599
number of classes of pivotal incomplete sets	53	931	3678	2240	168	1	7071

Table 5.

This table gives also the numbers of S-bases and S-pivotal incomplete sets of each rank of B_1, because each class of functions of B_1 contains a symmetric function.

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REZIME

SIMETRIČNE FUNKCIJE JEDNOG PRETPUNOG SKUPA TROZNAČNE LOGIKE

Miyakawa je dokazao da potpun skup $\mathrm{Pol}\begin{pmatrix}0&1&2&1&0&1&2\\0&1&2&0&1&2&1\end{pmatrix}$ sadrži 54 klasa funkcija. U ovom radu je pokazano da svaka klasa funkcija sadrži neku simetričnu funkciju.