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CR-SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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ABSTRACT

The notion of CR-submanifolds of a Kaehler manifold was introduced by A.Bejancu [1]. Later, CR-submanifolds of a Sasakian manifold were studied by M.Kobayashi [7]. In this paper, we shall study some properties of a D-totally geodesic and $D^{\perp}$-totally geodesic CR-submanifold of a Sasakian manifold. We also study the Ricci tensor and scalar curvature of D-minimal and $D^{\perp}$-minimal CR-submanifold of a Sasakian space form.

1. INTRODUCTION

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with structure tensors (ϕ,ξ,η,g) satisfying

(1.1) $\phi^2 = -1 + \eta\otimes\xi,\ \phi\xi = 0,\ \eta\circ\phi = 0,\ \eta(\xi) = 1$,

(1.2) $g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),\ g(X,\xi) = \eta(X)$,

for any vector fields X and Y on $\bar{M}$.

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$\bar{M}$ is called a Sasakian minifold if

(1.3) $(\bar{\nabla}_X\phi)Y = \eta(Y)X - g(X,Y)\xi$

(1.4) $\bar{\nabla}_X\xi = \phi X$

where $\bar{\nabla}$ is the Riemannian connection determined by g.

DEFINITION. *An m-dimensional Riemannian submanifold M of a Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D: p \to D_p \subset T_pM$ such that*

(i) D_p is invariant under ϕ i.e. $\phi D_p \subset D_p$, for each $p \in M$,

(ii) the orthogonal complementary distribution $D^\perp: p \to$ $\to D_p^\perp \subset T_pM$ of the distribution D on M is totally real, i.e. $\phi D_p^\perp \subset T_p^\perp M$, where T_pM and $T_p^\perp M$ are the tangent space and the normal space of M at p.

We call D (resp. $D^\perp$) the horizontal (resp. vertical) distribution. Moreover the pair $(D, D^\perp)$ is called ξ-horizontal (resp. ξ-vertical), if $\xi_p \in D_p$ (resp. $\xi_p \in D_p^\perp$) for each $p \in M$, [7].

Let the orthogonal complement of $\phi D^\perp$ in $T^\perp M$ be μ. Then we have

$$T_pM = D_p \oplus D_p^\perp, \quad T_p^\perp M = \phi D_p^\perp \oplus \mu_p .$$

It is obvious that

$$\phi\mu_p = \mu_p .$$

The distribution: $p \to T_p^\perp M$ on M is denoted by ν. Thus $\nu = \phi D^\perp \oplus \mu$.

The distribution D (resp. $D^\perp$) can be defined by a projector P (resp. Q) which satisfies the conditions

$$P^2 = P, \quad Q^2 = Q, \quad PQ = QP = =, \quad g(P,Q) = 0$$

The Gauss and Weingarten formulae are given by

(1.5) $\bar{\nabla}_XY = \nabla_XY + h(X,Y)$, (1.6) $$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N ,$$

where ∇ is the Riemannian connection on M, $\nabla^{\perp}$ is the connection on the normal bundle induced by ∇ and h is the second fundamental form of the immersion satisfying

(1.7) $$g(A_N X, Y) = g(h(X,Y), N) .$$

For a vector field N in the normal bundle, we put

(1.8) $$\phi N = BN + CN ,$$

where BN (resp. CN) is the vertical (resp. normal) part of ϕN.

The equation of Gauss is given by

(1.9) $$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z), h(Y,W)) - g(h(Y,Z), h(X,W))$$

where $\bar{R}$ (resp. R) is the curvature tensor of $\bar{M}$(resp. M) .

Calculating $(\bar{\nabla}_X \phi)Y$ in two different ways and comparing the horizontal, vertical and normal part, M.Kobayashi obtained the following [7].

(1.10) $$P\nabla_X \phi PY - PA_{\phi QY} X = \phi P\nabla_X Y + \eta(Y)PX - g(X,Y)P\xi ,$$

(1.11) $$Q\nabla_X \phi PY - QA_{\phi QY} X = Bh(X,Y) + \eta(Y)QX - g(X,Y)Q\xi ,$$

(1.12) $$h(X,\phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y + Ch(X,Y) .$$

2. D-TOTALLY GEODESIC AND $D^{\perp}$-TOTALLY GEODESIC CR-SUBMANIFOLD

DEFINITION. *A CR-submanifold* M *of a Sasakian manifold* $\bar{M}$ *is said to be* D*-totally geodesic (resp.* $D^{\perp}$*-totally geodesic) if* $h(X,Y) = 0$ *for* $X,Y \in D$ *(resp.* $h(Z,W) = 0$ *for* $W,Z \in D^{\perp}$*).*

PROPOSITION (2.1) *Let M be a CR-submanifold of a Sasakian manifold $\bar{M}$. Then M is D-totally geodesic if and only if $A_N X \in D^{\perp}$ for each $X \in D$, $N \in \nu$.*

P r o o f. Let M be D-totally geodesic. Then $h(X,Y)=0$ for $X,Y \in D$. Now for $N \in \nu$, we have

$$0 = g(h(X,Y),N) = g(A_N X,Y) ,$$

that is, $A_N X \in D^{\perp}$.

Conversely, suppose $A_N X \in D^{\perp}$. Then for $X,Y \in D$ we have

$$0 = g(A_N X,Y) = g(h(X,Y),N) ,$$

that is, $h(X,Y) = 0$, for $X,Y \in D$ which implies that M is D-totally geodesic.

The following is direct.

PROPOSITION (2.2). *Let M be a R-submanifold of a Sasakian manifold $\bar{M}$. Then D is $D^{\perp}$-totally geodesic if and only if $A_N X \in D$ for each $X \in D^{\perp}$ and $N \in \nu$.*

Now we shall prove

PROPOSITION (2.3) *Let M be a $D^{\perp}$-totally geodesic ξ-horizontal CR-submanifold of a Sasakian manifold $\bar{M}$. Then we have*

(i) $\phi A_N X = A_{\phi N} X$,

(ii) $\phi \nabla^{\perp}_X N = \nabla^{\perp}_X \phi N \in \mu$,

(iii) $\nabla^{\perp}_X N \in \mu$.

for any vector fields $X \in D^{\perp}$ and $N \in \mu$.

P r o o f. From (1.3) we get

$$(\bar{\nabla}_X \phi)N = \eta(N)X - g(X,N)\xi = 0$$

which gives that

$$\bar{\nabla}_X \phi N = \phi \bar{\nabla}_X N .$$

Now using the Weingarten formula, we have

$$\phi A_N X - \phi \nabla^{\perp}_X N = A_{\phi N} X - \nabla^{\perp}_X \phi N .$$

Now, from Proposition (2.2), $A_N X \in D$ for $X \in D$ and $N \in \mu$. Hence

$$\phi A_N X \in D,\ A_{\phi N} X \in D. \text{ Also } \phi \nabla^{\perp}_X N \in D^{\perp} \oplus \mu,\ \nabla^{\perp}_X \phi N \in \phi D^{\perp} \oplus \mu .$$

Consequently, $\phi A_N X = A_{\phi N} X$,

and $\phi \nabla^{\perp}_X N = \nabla^{\perp}_X \phi N \in \mu$.

Further, $\phi \nabla^{\perp}_X N \in \mu$ and $\phi\mu = \mu$ which gives that $\nabla^{\perp}_X N \in \mu$.

DEFINITION. *The horizontal (resp. vertical) distribution* D *(resp.* $D^{\perp}$*) is said to be parallel* [1] *with respect to the connection on* M *if* $\nabla_X Y \in D$ *(resp.* $\nabla_Z W \in D^{\perp}$ *) for any vector fields* $X, Y \in D$ *(resp.* $W, Z \in D^{\perp}$*).*

We shall prove

PROPOSITION (2.4) *Let* M *be a* ξ*-vertical CR-submanifold of a Sasakian manifold* $\bar{M}$*. Then the distribution* $D^{\perp}$ *is parallel with respect to the connection on* M*, if and only if* $A_N Z \in D^{\perp}$ *for each* $Z \in D^{\perp}$ *and* $N \in \nu$.

P r o o f. Let $W, Z \in D^{\perp}$. Then using the Gauss and Weingarten formulae, we have

$$-A_{\phi W} Z + \nabla^{\perp}_Z \phi W = \phi \nabla_Z W + \phi h(W,Z) + \eta(W) Z - g(W,Z)\xi .$$

Taking the inner product with $Y \in D$, we have

$$-g(A_{\phi W} Z, Y) = g(\phi \nabla_Z W, Y) + \eta(W) g(Z,Y) - g(W,Y)\eta(Y) ,$$

$$-\ g(A_{\phi W} Z, Y) = -\ g(\nabla_Z W, \phi Y) .$$

Therefore, $\nabla_Z W \in D$ if and only if $A_N Z \in D^{\perp}$ for all $Z \in D^{\perp}$,

$N \in \phi D^{\perp}$ whereby the result is proved.

Now we shall prove

PROPOSITION (2.5.) *Let M be a ξ-vertical CR-submanifold of a Sasakian manifold* $\bar{M}$. *Then*

$$\phi Ch(X,Y) = Ch(\phi X,Y) = Ch(X,\phi Y) \tag{2.1}$$

for $X,Y \in D$.

P r o o f. Using (1.11), we have, for $X,Y \in D$.

$$Q\nabla_{\phi X}\phi Y = Bh(\phi X,Y) - g(\phi X,Y)Q\xi \ . \tag{2.2}$$

Again, interchanging X and Y in (1.11), we get

$$Q\nabla_Y\phi X = Bh(X,Y) - g(X,Y)Q\xi \ .$$

Replacing X by ϕX in the above equation, we obtain

$$Q\nabla_Y(\phi^2 X) = Bh(\phi X,Y) - g(\phi X,Y)Q\xi$$

$$Q\nabla_Y X = - Bh(\phi X,Y) + g(\phi X,Y)Q\xi \tag{2.3}$$

adding (2.2) and (2.3), we have

$$\nabla_Y X + \nabla_{\phi X}\phi Y \in D \ .$$

Now from (1.12), we have

$$h(X,\phi Y) = \phi Q(\nabla_X Y) + Ch(X,Y) \ ,$$

which, on replacing X and Y by ϕX and ϕY, gives

$$-h(\phi X,Y) = \phi Q(\nabla_{\phi X}\phi Y) + Ch(\phi X,\phi Y). \tag{2.4}$$

Also, interchanging X and Y in (1.12), we have

$$h(\phi X,Y) = \phi Q(\nabla_Y X) + Ch(X,Y). \tag{2.5}$$

Adding (2.4) and (2.5) and using the fact that $\nabla_Y X + \nabla_{\phi Y}\phi X \in D$, we have

$$Ch(\phi X,\phi Y) + Ch(X,Y) = 0,$$

or $$Ch(\phi^2 X,\phi Y) + Ch(\phi X,Y) = 0 \ ,$$

and consequently, $Ch(X,\phi Y) = Ch(\phi X,Y)$.

Again from (1.11), we have

$$Q(\nabla_X \phi Y) = Bh(X,Y) - g(X,Y)Q\xi$$

or, $$Q(\nabla_X \phi^2 Y) = Bh(X, \phi Y) - g(X,\phi Y)Q\xi$$

and hence

$$Q\nabla_X Y = - Bh(X,\phi Y) + g(X,\phi Y)Q\xi \ .$$

Using the above equation in (1.12), we have

$$\begin{aligned} h(X,\phi Y) &= \phi(Q\nabla_X Y) + Ch(X,Y) \ , \\ &= \phi(-Bh(X,\phi Y) + g(X,\phi Y)Q\xi) + Ch(X,Y) . \\ &= - \phi Bh(X,\phi Y) + Ch(X,Y) \ . \end{aligned}$$

Applying ϕ on both sides, we get

$$\phi h(X,\phi Y) = Bh(X,\phi Y) + \phi Ch(X,Y) \ .$$

Then using (1.8) in the above equation, we get

$$Ch(X,\phi Y) = \phi Ch(X,Y)$$

which completes the proof of the proposition.

M.Kobayashi [7] has shown the

PROPOSITION. *Let* M *be a CR-submanifold of a Sasakian manifold* $\bar{M}$. *If* M *is* ξ*-horizontal, then the distribution* D *is integrable if and only if*

$$h(X,\phi Y) = h(Y,\phi X) \quad \textit{for all} \quad X,Y \in D.$$

Using the above proposition, we have

PROPOSITION (2.6) *Let M be a ξ-horizontal CR-submanifold of a Sasakian manifold $\bar{M}$. Then the horizontal distribution D is parallel if and only if*

$$h(X,\phi Y) = h(\phi X,Y) = \phi h(X,Y) \quad \text{for} \quad X,Y \in D. \tag{2.6}$$

P r o o f. Since every parallel distribution is involutive the first equality in (2.6) follows immediately.

Now since $\nabla_X \phi Y \in D$ for $X,Y \in D$, using (1.11), we have

$$Bh(X,Y) = 0$$

Therefore, from $\phi h(X,Y) = Bh(X,Y) + Ch(X,Y)$, we have

$$\phi h(X,Y) = Ch(X,Y).$$

The converse part follows from equation (1.12).

3. RICCI TENSOR AND SCALAR CURVATURE OF D-MINIMAL OR D-MINIMAL CR-SUBMANIFOLD OF A SASAKIAN SPACE FORM

Let $\{E_o = \xi,\ E_1,\ldots,E_{m-1}\}$ be a local field of orthonormal frames on M such that in case when M is ξ-horizontal $\{E_o = \xi,\ E_1,\ldots,E_p, E_{p+1} = \phi E_1,\ E_{2p} = \phi E_p\}$ is a local frame field on D and $\{F_1,\ldots,F_q\}$ is a local frame field on $D^\perp$.

Let M be an ξ-horizontal CR-submanifold of $\bar{M}$. The mean curvature vector field H of M in $\bar{M}$ is defined by

$$H = \frac{1}{m}\left\{ \sum_{i=1}^{2p+1} h(E_i,E_i) + \sum_{k=1}^{q} h(F_k,F_k) \right\}. \tag{3.1}$$

If $H = 0$, then M is said to be minimal. Now we shall define

$$H_D = \frac{1}{2p+1} \sum_{i=1}^{2p+1} h(E_i,E_i) , \tag{3.2}$$

$$(3.3)\qquad H_{D^\perp} = \frac{1}{q}\sum_{k=1}^{q} h(F_k,F_k) .$$

If $H_D = 0$, then the CR-submanifold M is said to be D-minimal, and if $H_{D^\perp} = 0$, then it is said to be $D^\perp$-minimal. Similar definitions can be given for ξ-vertical CR-submanifolds.

Now, suppose $\bar{M}(C)$ to be a Sasakian space form. Then the curvature tensor of $\bar{M}(C)$ is given by [7].

$$(3.4)\qquad \bar{R}(X,Y)Z = \frac{(C+3)}{4}\{g(Y,Z)X - g(X,Z)Y\} - \frac{(C-1)}{4}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi - g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y + 2g(\phi X,Y)\phi Z\}.$$

Let M be a CR-submanifold of a Sasakian space form $\bar{M}(C)$. Then the equation of Gauss is given by

$$(3.5)\qquad R(X,Y,Z,W) = \frac{(C+3)}{4}\{g(Y,Z)g(X,W) - g(Z,X)g(Y,W)\} - \frac{(C-1)}{4}\{\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W) + \eta(X)\eta(W)g(Y,Z) - \eta(Y)\eta(W)g(X,Z) - g(\phi PY,Z)g(\phi PX,W) + g(\phi PX,Z)g(\phi PY,W) + 2g(\phi PX,Y)g(\phi PZ,W)\} + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) .$$

Let $X,Y \in D$, $Z,W \in D^\perp$ and U,V be any vector field tangent to M. The Ricci tensor and the scalar curvature are given by

$$(3.6)\qquad S(U,V) = \sum_{i=1}^{2p+1} g(R(E_i,U)V,E_i) + \sum_{k=1}^{q} g(R(F_k,U)V,F_k) ,$$

$$(3.7)\qquad \rho = \sum_{i=1}^{2p+1} S(E_i,E_i) + \sum_{k=1}^{q} S(F_k,F_k) .$$

Also, we define

$$S_D(U,V) = \sum_{i=1}^{2p+1} g(R(E_i,U)V,E_i) \ , \tag{3.8}$$

$$S_{D^\perp}(U,V) = \sum_{k=1}^{q} g(R(F_k,U)V,F_k) \ , \tag{3.9}$$

$$\rho_{DD} = \sum_{i=1}^{2p+1} S_D(E_i,E_i), \quad \rho_{DD^\perp} = \sum_{k=1}^{q} S_D(F_k,F_k) \ , \tag{3.10}$$

$$\rho_{D^\perp D} = \sum_{i=1}^{2p+1} S_{D^\perp}(E_i,E_i), \quad \rho_{D^\perp D^\perp} = \sum_{k=1}^{q} S_{D^\perp}(F_k,F_k), \tag{3.11}$$

Now for $X,Y \in D$ and $Z,W \in D^\perp$ we get

$$S_D(X,Y) = \frac{1}{2}\{(C+3)p+(C-1)\}g(X,Y) - \frac{1}{2}(C-1)(p+1)\eta(X)\eta(Y)$$
$$+ g((2p+1)H_D,h(X,Y)) - \sum_{i=1}^{2p+1} g(h(E_i,Y),h(X,E_i)), \tag{3.12}$$

$$S_D(X,Z) = g((2p+1)H_D,h(X,Z)) - \sum_{i=1}^{2p+1} g(h(E_i,Z),h(X,E_i)), \tag{3.13}$$

$$S_D(Z,W) = \frac{1}{4}\{(C+3)(2p+1) - (C-1)\}g(Z,W)$$
$$+ g((2p+1)H_D,h(Z,W)) - \sum_{i=1}^{2p+1} g(h(E_i,Z),h(W,E_i)), \tag{3.14}$$

$$S_{D^\perp}(Z,W) = \frac{1}{4}(C+3)(q-1)g(Z,W) + g(qH_{D^\perp},h(Z,W))$$
$$- \sum_{k=1}^{q} g(h(F_k,W),h(Z,F_k)) \ , \tag{3.15}$$

$$S_{D^\perp}(X,Z) = g(qH_{D^\perp},h(X,Z)) - \sum_{k=1}^{q} g(h(F_k,Z),h(X,F_k)), \tag{3.16}$$

$$S_{D^\perp}(X,Y) = \frac{1}{4}(C+3)q\ g(X,Y) - \frac{1}{4}(C-1)q\eta(X)\eta(Y)$$
$$+ g(qH_{D^\perp},h(X,Y)) - \sum_{k=1}^{q} g(h(F_k,Y),h(X,F_k)). \tag{3.17}$$

Now we have

(3.18) $$\rho_{DD} = \sum_{i=1}^{2p+1} S_D(E_i,E_i)$$

$$= p(pC+3p+C+1) - (2p+1)^2 g(H_D,H_D)$$

$$- \sum_{i,j=1}^{2p+1} g(h(E_j,E_i),h(E_i,E_j)) ,$$

(3.19) $$\rho_{D^\perp D} = \sum_{i=1}^{2p+1} S_{D^\perp}(E_i,E_i)$$

$$= \frac{1}{2} q(pC+3p+2) + q(2p+1) g(H_{D^\perp},H_D)$$

$$- \sum_{i=1}^{2p+1} \sum_{k=1}^{q} g(h(F_k,E_i),h(E_i,F_k)) ,$$

(3.20) $$\rho_{DD^\perp} = \sum_{k=1}^{q} S(F_k,F_k)$$

$$= \frac{1}{2} q(pC+3p+2) + (2p+1)q\ g(H_D,H_{D^\perp})$$

$$- \sum_{i=1}^{2p+1} \sum_{k=1}^{q} g(h(F_k,E_i),h(E_i,F_k)) ,$$

(3.21) $$\rho_{D^\perp D^\perp} = \sum_{k=1}^{q} S_{D^\perp}(F_k,F_k)$$

$$= \frac{1}{4}(C+3)(q-1)q + q^2 g(H_{D^\perp},H_{D^\perp})$$

$$- \sum_{h,k=1}^{q} g(h(F_k,F_k),h(F_k,F_k)) .$$

From (3.19) and (3.20), we observe that

$$\rho_{DD^\perp} = \rho_{D^\perp D}$$

Now we shall prove

PROPOSITION (3.1) *Let* M *be a* D-*minimal* ξ-*horizontal CR-submanifold of a Sasakian space form* $\bar{M}(C)$. *Then,*

(a) $$S_D(X,Y) - \frac{1}{2}\{(C+3)p + (C-1)\}g(X,Y) + \frac{1}{2}(C-1)(p+1)\eta(X)\eta(Y)$$

is negative semi-definite for $X, Y \in D$.

(a′) $$S_D(Z,W) - \frac{1}{4}\{(C+3)(2p+1) - (C-1)\}g(Z,W)$$

is negative semi-definite for $Z, W \in D^{\perp}$.

(b) $\rho_{DD} \leq p(pC + 3p + C + 1)$.

(b′) $\rho_{DD^{\perp}} \leq \frac{1}{2}(pC + 3p + 2)q$.

P r o o f. From (3.12) and (3.14), we have

$$S_D(X,Y) - \frac{1}{2}\{(C+3)p + (C-1)\}g(X,Y) + \frac{1}{2}(C-1)(p+1)\eta(x)\eta(Y)$$
$$= -\sum_{i=1}^{2p+1} g(h(E_i,Y),h(X,E_i))$$
$$S_D(Z,W) - \frac{1}{4}\{(C+3)(2p+1) - (C-1)\}g(Z,W)$$
$$= \sum_{i=1}^{2p+1} g(h(E_i,Z),h(W,E_i)).$$

Also from (3.18) and (3.20), we have

$$\rho_{DD} = p(pC + 3p + C + 1) - \sum_{i,j=1}^{2p+1} g(h(E_j,E_i),h(E_i,E_j)) ,$$

$$\rho_{DD^{\perp}} = \frac{1}{2}q(pC+3p+2) - \sum_{i=1}^{2p+1}\sum_{k=1}^{q} g(h(E_i,F_k),h(F_k,E_i)).$$

From these formulae, the theorem follows.

Also we have

PROPOSITION (3.2) *Let M be a $D^{\perp}$-minimal ξ-horizontal CR-submanifold of a Sasakian space form $\bar{M}(C)$. Then for $S_{D^{\perp}}$, $\rho_{D^{\perp}D^{\perp}}$ and $\rho_{D^{\perp}D}$ we have that*

(a) $S_{D^{\perp}}(Z,W) - \frac{1}{4}(C+3)(q-1)g(Z,W)$ *is negative semi-definite for* $Z,W \in D^{\perp}$.

(a′) $S_{D^{\perp}}(X,Y) - \frac{1}{4}(C+3)q\, g(X,Y) + \frac{1}{4}(C-1)q\, \eta(X)\eta(Y)$

is negative semi-definite for $X,Y \in D$.

(b) $\rho_{D^{\perp}D^{\perp}} \leq \frac{1}{4}(C+3)(q-1)q$.

(b′) $\rho_{D^{\perp}D} \leq \frac{1}{2}q(pC+3p+2)$.

PROPOSITION (3.3) *Let M be a D-minimal ξ-horizontal CR-submanifold of a Sasakian space form $\bar{M}(C)$. Then M is D-totally geodesic if and only if M satisfies one of the following conditions*

(a) $S_D(X,Y) = \frac{1}{2}\{(C+3)p + (C-1)\}g(X,Y)$

$+ \frac{1}{2}(C-1)(p+1)\eta(X)\eta(Y)$, *for* $X,Y \in D$.

(b) $\rho_{DD} = p(pC+3p+C+1)$.

PROPOSITION (3.4) *Let M be a $D^{\perp}$-minimal ξ-horizontal CR-submanifold of a Sasakian space form $\bar{M}(C)$. Then M is $D^{\perp}$-totally geodesic if and only if M satisfies one of the following conditions:*

(a) $K_M(Z,W) = \frac{1}{4}(C+3)$ *for* $Z,W \in D^{\perp}$ [*M.Kobayashi* [8] *th (3.5)*]

where $K_M(Z,W)$ *is the sectional curvature determined by orthonormal vectors Z and W.*

(b) $S_{D^{\perp}}(Z,W) = \frac{1}{4}(C+3)(q-1)g(Z,W)$, *for* $Z,W \in D^{\perp}$.

(c) $\rho_{D^{\perp}D^{\perp}} = \frac{1}{4}(C+3)(q-1)q$.

Now considering the formulae (3.12)-(3.21), we obtain the following result due to M.Kobayashi (c.f. [7], Th. 4.1).

THEOREM. *Let M be a minimal ξ-horizontal CR- submanifold of a Sasakian space form $\bar{M}(C)$. Then,*

(a) $$S - \frac{1}{4}\{(C+3)(m-1) + 2(C-1)\}g_{0}(P,P) - \frac{1}{4}\{(C+3)(m-1) - (C-1)\}g_{0}(Q,Q) + \frac{1}{4}(C-1)(m+1)\eta\cdot\eta$$

is negative semi-definite.

(b) $$\rho < \frac{1}{4}\{(C+3)(m-1) + 2(C-1)\}(2p+1) + \frac{1}{4}\{(C+3)(m-1) - (C-1)\}(m-2p-1) - \frac{1}{4}(C-1)(m+1).$$

P r o o f. By (3.18)-(3.21), we have

$$S(X+Z,\ Y+W) = S(X,Y) + S(Z,Y) + S(X,W) + S(Z,W)$$

$$= S_D(X,Y) + S_{D^\perp}(X,Y) + S_D(Y,Z) + S_{D^\perp}(Y,Z) + S_D(X,W) + S_{D^\perp}(X,W) + S_D(Z,W) + S_{D^\perp}(Z,W).$$

$$= \frac{1}{4}\{(C+3)(m-1) + 2(C-1)\}g(X,Y) + \frac{1}{4}\{(C+3)(m-1) - (C-1)\}g(Z,W) - \frac{1}{4}(C-1)(m+1)\eta(X)\eta(Y) + g(mH, h(X+Z,\ Y+W)) - \sum_{\ell=1}^{m} g(h(E_\ell, X+Z),\ h(E_\ell,\ Y+W)) .$$

Again

$$S(U,V) = S_D(U,V) + S_{D^\perp}(U,V)$$

$$\rho = \sum_{i=1}^{2p+1} S(E_i,E_i) + \sum_{k=1}^{q} S(F_k,F_k)$$

$$= \sum_{i=1}^{2p+1} S(E_i,E_i) + \sum_{i=1}^{2p+1} S_{D^\perp}(E_i,E_i) + \sum_{k=1}^{q} S_D(F_k,F_k)$$

$$+ \sum_{k=1}^{q} S_{D^\perp}(F_k,F_k)$$

$$= \rho_{DD} + \rho_{D^\perp D} + \rho_{DD^\perp} + \rho_{D^\perp D^\perp}$$

$$= \frac{1}{4}\{(C+3)(m-1) + 2(C-1)\}(2p+1) + \frac{1}{4}\{(C+3)(m-1)$$

$$- (C-1)\}(m-2p-1)$$

$$- \frac{1}{4}(C-1)(m+1) + g(mH,mH) - \sum_{i,j=1}^{2p+1} g(h(E_i,E_j),h(E_i,E_j))$$

$$- 2\sum_{k=1}^{q}\sum_{i=1}^{2p+1} g(h(E_i,F_k),h(E_i,F_k))$$

$$- \sum_{j,k=1}^{q} g(h(F_j,F_k),\ h(F_j,F_k)) .$$

from which the theorem follows.

REMARK. Similar results can be obtained for a ξ-vertical CR-submanifold of a Sasakian space form.

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REZIME

CR-PODMNOGOSTRUKOSTI SASAKIANOVE MNOGOSTRUKOSTI

Pojam CR-podmnogostrukosti Kaehlerove mnogostrukosti je uveden od A.Bejancua [1]] . Kasnije,CR-podmnogostrukost Sasakianove mnogostrukosti je izučavao M.Kobayashi [7] . U ovom radu su izučene neke osobine D-totalno geodezijske i D^1-totalno geodezijske CR-podmnogostrukosti Sasakianove mnogostrukosti . Takodje je izučen tenzor Riccia i skalarna krivina D-minimalne i D^1-minimalne CR-mnogostrukosti forme Sasakianovog prostora .