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REMARKS ON ALTERNATING SYMMETRIC n-QUASIGROUPS

Wiesław A. Dudek

Institute of Mathematics, Pedagogical University
Al. Zawadzkiego 13/15, 42-200 Częstochowa, Poland

ABSTRACT

Alternating symmetric n-quasigroups were introduced and described by Z. Stojaković in [9] as a generalization of semisymmetric (binary) quasigroups.

In this note we generalize a part of the results obtained by Z. Stojaković and give some connections between alternating symmetric and totally symmetric n-quasigroups. Special elements in alternating symmetric n-quasigroups are described, too.

1. INTRODUCTION

We shall use the following abreviated notation:

$$f(x_1,\dots,x_k,x_{k+1},\dots,x_{k+s},x_{k+s+1},\dots,x_n) =$$

$$= f(x_1^k, \overset{(s)}{x}, x_{k+s+1}^n),$$

whenever $x_{k+1} = x_{k+2} = \dots = x_{k+s} = x$ (x_j^i is the empty symbol for $i > j$ and for $i > n$, also $\overset{(0)}{x}$ is the empty symbol.

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To avoid repetitions we assume throughout the whole text that $n > 2$.

By S_n we denote the symmetric group of degree n and by A_n its alternating subgroup.

If $\sigma \in S_n$, then $x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(j)}$ we denote by $\{x_{\sigma(k)}\}_{k=i}^{j}$ or by $x_{\sigma(i)}^{\sigma(j)}$. If $i > j$, then $x_{\sigma(i)}^{\sigma(j)}$ is considered empty.

If (G,f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup (G,f^{σ}) defined by

$$f^{\sigma}(\{x_{\sigma(i)}\}_{i=1}^{n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}$$

is called a *σ-parastrophe* of (G,f).

If $\sigma, \tau \in S_{n+1}$, then $(f^{\sigma})^{\tau} = f^{\sigma\tau}$ and

$$f(\{x_{\sigma(i)}\}_{i=1}^{n}) = x_{\sigma(n+1)} \Leftrightarrow f^{\tau}(\{x_{\sigma\tau(i)}\}_{i=1}^{n}) = x_{\sigma\tau(n+1)} .$$

If (G,f) is an n-quasigroup and $\sigma \in S_{n+1}$ such that $f = f^{\sigma}$, then the permutation σ is called an *autoparastrophism* of f. The set of all autoparastrophisms of f is a subgroup of S_{n+1} and is denoted by $\pi(f)$.

An n-groupoid (G,f) is called *(i,j)-commutative* iff

$$f(x_1^{i-1}, x_i, x_{i+1}^{j-1}, x_j, x_{j+1}^{n}) = f(x_1^{i-1}, x_j, x_{i+1}^{j-1}, x_i, x_{j+1}^{n})$$

for all $x_1^n \in G$. It is clear that an n-quasigroup (G,f) is (i,j)-commutative iff $f = f^{(i,j)}$. A commutative n-groupoid is (i,j)-commutative for every pair $(i,j) \in N_n \times N_n$, where $N_n = \{1,2,\ldots,n\}$.

An n-groupoid (G,f) is called *totally symmetric* iff

$$f(\{x_{\sigma(i)}\}_{i=1}^{n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1} \tag{1}$$

for every $\sigma \in S_{n+1}$.

An n-groupoid (G,f) is called *an alternating symmetric n-groupoid* (in short: *AS-n-groupoid*) iff (1) holds for every $\sigma \in A_{n+1}$.

As it is well-known any set of (n-1)-place functions over a given set A which is closed under superposition of these functions could by isomorphically represented as an n-groupoid (G,f) such that

$$f(f(x_1^n),y_2^n) = f(x_1,f(x_2,y_2^n),f(x_3,y_2^n),f(x_3,y_2^n),\ldots,f(x_n,y_2^n))$$

for all $x_1^n, y_2^n \in G$. An n-groupoid with this property is called *a Menger* n-*groupoid* or *a Dicker* n-*groupoid* (in the other terminology: *a Menger algebra of rank* n-1 [7]).

An n-quasigroup (G,f) is *medial* iff

$$f(f(x_{11}^{1n}),f(x_{21}^{2n}),\ldots,f(x_{n1}^{nn}) =$$

$$= f(f(x_{11}^{n1}),f(x_{12}^{n2}),\ldots,f(x_{1n}^{nn}))$$

for all $x_{ij} \in G$, $i,j \in N_n$.

Other notions and definitions are standard and can be found for example in [1].

2. FUNDAMENTAL PROPERTIES

From the above definitions it follows that every totally symmetric n-groupoid is also alternating symmetric. But there are alternating symmetric n-groupoids which are not totally symmetric, which follows from [6] where D.G. Hoffman has proved that for every $m > n$, $p \geq 2$, and every subgroup $H \subseteq S_{n+1}$ there exists an n-quasigroup (G,f) of the order mp such that $\pi(f) = H$.

For n = 2 from the definition it follows that a groupoid $(G,\cdot)$ is alternating symmetric iff $(\cdot) = (\cdot)^{(1,2,3)} = (\cdot)^{(1,3,2)}$ i.e. iff

$$xy = z \Leftrightarrow yz = x \Leftrightarrow zx = y.$$

These equivalences imply that a groupoid $(G,\cdot)$ is alternating symmetric iff the identities

$$y(xy) = x \text{ and } (yx)y = x$$

hold.

A binary groupoid satisfying these identities is called semisymmetric and it must be a quasigroup. The analogous result holds for n-ary groupoids, i.e. every alternating symmetric n-groupoid is an n-quasigroup. *An n-quasigroup (G,f) is alternating symmetric iff* $f = f^{\sigma}$ *for all permutations* $\sigma \in A_{n+1}$, *i.e. iff* $A_{n+1} \subseteq \pi(f)$.

Let Γ be the generating set of the group A_{n+1}. From the fact that every alternating symmetric n-groupoid is an alternating symmetric n-quasigroup and the results of [9] it follows that an n-groupoid (G,f) is alternating symmetric iff $f = f^{\sigma}$ for all $\sigma \in \Gamma$, i.e. iff

$$f(x_{\sigma(1)}^{\sigma(k-1)}, f(x_1^n), x_{\sigma(k+1)}^{\sigma(n)}) = x_{\sigma(n+1)} \tag{2}$$

for all $x_1^n \in G$ and for every $\sigma \in \Gamma$ such that $\sigma^{-1}(n+1) = k \in N_n$.

Starting from different generating sets of A_{n+1}, we obtain the different systems of identities defining the class of all alternating symmetric n-groupoids. Hence, we have the following theorem.

Theorem 1. *The class of all alternating symmetric n-groupoids (i.e. n-quasigroups) is equationally definable.*

This class is uniquelly definable by any of the systems of n-1 identities which were given in Corollary 1 from [9], and analogously other (equivalent) systems of defining identities can be constructed.

In [9] it is proved that every AS-n-group (G,f) has the form

$$f(x_1^n) = x_1 + x_2 + x_3 + \ldots + x_n + c,$$

where c is a fixed element of G and (G,+) is a Boolean group. This result follows also from our Theorem 1 and from Corollary 9 from [3]. Indeed, since $\Gamma = \{(1,2,3),(1,2,4),\dots,(1,2,n+1)\}$ generates A_{n+1}, then

$$(3) \qquad f(x_1^{i-1}, f(x_1^n), x_i, x_{i+2}^n) = x_{i+1}$$

for every $i \in N_{n-1}$. But Γ contains the cycle (1,2,n+1), hence

$$f(y, \overset{(n-2)}{x}, \bar{x}) = y \Leftrightarrow f(x, y, \overset{(n-3)}{x}, \bar{x}) = y .$$

The first identity holds in every n-group, but the second only in commutative (see Corollary 9 from [3]). Thus $f(x_1^n) = x_1 + \dots + x_n + c$, where $c \in G$ and (G,+) is a commutative group. The condition (3) shows that (G,+) is Boolean. From (3) follows also that $f(x,x,\dots,x) = \bar{x}$ for every $x \in G$. On the other hand it is easy to verify that $c = f(\bar{a},\bar{a},\dots,\bar{a})$ if $x+y = f(y, \overset{(n-2)}{a}, y)$. Hence, if n is even, then

$$x + c = f(x, \overset{(n-2)}{a}, f(\overset{(n)}{\bar{a}})) = f(x \overset{(n-1)}{\bar{a}}) =$$

$$= f(x, f(\overset{(n)}{a}), \overset{(n-2)}{\bar{a}}) = f(x, a, f(\overset{(n-1)}{a}, \bar{a}), \overset{(n-3)}{\bar{a}}) =$$

$$= f(x, \overset{(2)}{a}, \overset{(n-3)}{\bar{a}}) = f(x, \overset{(2)}{a}, f(\overset{(n)}{a}), \overset{(n-4)}{\bar{a}}) =$$

$$= f(x, \overset{(4)}{a}, \overset{(n-5)}{\bar{a}}) = \dots = f(x, \overset{(n-2)}{a}, \bar{a}) = x .$$

So we have the following theorem.

Theorem 2. *Let (G,f) be an n-group. Then (G,f) is alternating symmetric iff there exists a Boolean group (G,+) such that*

$$f(x_1^n) = x_1 + x_2 + \dots + x_n$$

if n is even, or

$$f(x_1^n) = x_1 + x_2 + \dots + x_n + c,$$

where c *is a fixed element from* G, *if* n *is odd.*

Another generalizations of semisymmetric groupoids were introduced in [8] and [5]. In [5] so-called i-*permutable* n-*groupoids* are described, i.e. n-groupoids with the identity $f(x_1^{i-1}, f(x_1^n), x_i^{n-1}) = x_n$. For i = 1 we obtain so-called cyclic n-quasigroups from [8] (see also [5]).

Since an n-quasigroup (G,f) is i-permutable iff $f = f^{\sigma}$, where σ is the cycle (i,i+1,i+2,...,n+1), and this cycle is even iff n-i is odd, we obtain

Proposition 1. *An* AS-n-*groupoid is* i-*permutable for every* $i \in N_n$ *such that* n-i *is odd. In particular, if* n *is even, then any* AS-n-*groupoid is a cyclic* n-*quasigroup.*

3. CONNECTIONS BETWEEN ALTERNATING AND TOTALLY SYMMETRIC n-GROUPOIDS

Let $i \in N_n$ and let π_i be the transposition (i,n+1), then the parastrophe f^{π_i} of an n-quasigroup (g,f) defined by

$$f^{\pi_i}(x_1^{i-1}, x_{n+1}, x_{i+1}^n) = x_i \Leftrightarrow f(x_1^n) = x_{n+1}$$

is called *the* i-*th inverse operation for* f.

Since $\pi_{i+1}\pi_i = (i,i+1,n+1)$ and $\pi_i\pi_{i+1} = (i,n+1,i+1)$ for all $i \in N_{n-1}$ and $\Gamma_1 = \{(1,2,n+1),(2,3,n+1),\dots,(n-1,n,n+1)\}$ $\Gamma_2 = \{(1,n+1,2),(2,n+1,3),\dots,(n-1,n,n+1)\}$ are generating sets of A_{n+1}, then we obtain the following characterization of AS-n-quasigroups (AS-n-groupoids).

Proposition 2. *An* n-*quasigroup* (G,f) *is* AS *iff*

$$f^{\pi_{i+1}\pi_i} = f \text{ or } f^{\pi_i\pi_{i+1}} = f$$

for every $i \in N_{n-1}$.

Proposition 3. *An* AS-n-*quasigroup* (G,f) *is totally symmetric iff* $f^{(i,j)} = f$ *for some* $1 \le i < j \le n+1$.

Since the permutation $(i,j)\sigma$ is odd for every $\sigma \in A_{n+1}$ and for all $1 \le i < j \le n+1$, then as a simple consequence of the above results we obtain

Corollary 1. *An* AS-n-*groupoid* (G,f) *is totally symmetric iff it is* (i,j)*-commutative for some* $i,j \in N_n$. *In paricular, every commutative* AS-n-*quasigroup is totally symmetric.*

As it is well-known (see for example [1], p. 47), for every medial n-quasigroup (G,f) there exists an Abelian group $(G,\cdot)$ and its automorphisms $\theta_1,\theta_2,\dots,\theta_n$ such that $\theta_i\theta_j = \theta_j\theta_i$ for $i \neq j$ and an element $b \in G$ such that $f(x_1^n) = \theta_1x_1\theta_2x_2,\dots,\theta_nx_nb$. If this n-quasigroup is AS, then $f^{(i,i+1,i+2)} = f$ for every $i \in N_{n-2}$, which implies $\theta_1 = \theta_2 = \theta_3 = \dots = \theta_n = \theta$. Hence (G,f) is a commutative n-quasigroup and has the form $f(x_1^n) = \theta(x_1x_2x_3 \dots x_n)b$. By Corollary 2 it is also totally symmetric.

So we have the following theorem.

Theorem 3. *Every medial* AS-n-*groupoid* (G,f) *is totally symmetric* n-*quasigroup and has the form*

$$f(x_1^n) = \theta(x_1x_2x_3 \dots x_nc),$$

where $(G,\cdot)$ *is an Abelian group,* θ *its automorphism and* c *a fixed element from* G.

Now we consider retracts of totally symmetric n-groupoids.

It is obvious that if (G,f) is a totally symmetric n-groupoid, then every m-ary retract of (G,f), where $2 \le m \le n$, is a totally symmetric m-quasigroup.

It is not dificult to see that the analogous result holds for alternating symmetric n-groupoids.

Proposition 4. *Let* (G,f) *be an* AS-n-*groupoid and* *let* $2 \le m \le n$. *Then every* m-*ary retract of* (G,f) *is an* AS-m-*-quasigroup.*

Since every (n-2)-ary retract of an AS-n-quasigroup is totally symmetric [9], then from the above propositions immediately follows

Corollary 2. *Any* m-*ary* ($2 \le m \le n-2$) *retract of an* AS-n-*groupoid* (G,f) *is a totally symmetric* m-*quasigroup.*

4. AS-n-QUASIGROUPS WITH SPECIAL ELEMENTS

Let (G,f) be a given an n-groupoid. The solution (if it exists) $x \in G$ of the equation

$$f(\overset{(i-1)}{a},x,\overset{(n-i)}{a}) = a \tag{4}$$

is called *the* i-*th skew element to* a and is denoted by $\hat{a}^{(i)}$. If (G,f) is an n-quasigroup, then this solution always exists and it is uniquely determined. If (G,f) is an n-group, then $\hat{a}^{(i)} = \hat{a}^{(j)}$ for all $i,j \in N_n$ and for every $a \in G$. The similar result holds in AS-n-quasigroups.

If $\hat{a}^{(i)} = \hat{a}^{(j)}$ for every $i,j \in N_n$, then the i-th skew element to a is denoted by $\bar{a}$ and is called (as in the n-group theory) *the skew element to* a. From our Theorem 2 follows that *in an* AS-n-*group* (G,f), *where* n *is even, there exists an element* e *(the unity of the corresponding Boolean group) which is skew to every* $a \in G$. *If* n *is odd, then* $\bar{a} \neq \bar{c}$ *for all* $a \neq c$.

Proposition 5. *In an* AS-n-*groupoid every element has a skew element.*

Proof. Since an AS-n-groupoid is an n-quasigroup, then the equation (4) has a solution for every $a \in G$ and for every $i \in N_n$. But cycles $(1,n+1,n)$ and $(i,i+1,n+1)$ are even permutations for any $i \in N_{n-1}$, hence (4) implies $f(a,a,\ldots,a) = \hat{a}^{(i)}$ for all $i \in N_n$, which completes our proof.

Corollary 3. *Let* (G,f) *be an* AS-n-*groupoid. Then* $f(a,a,\dots,a) = \bar{a}$ *for any* $a \in G$.

Now we consider AS-n-groupoids in which the identity

$$f(\overset{(i)}{e}, x, \overset{(n-i-1)}{e}) = x \tag{5}$$

holds for some $e \in G$, $i \in N_{n-1}$ and for all $x \in G$. *If (5) holds for every* $i = 0,1,2,\dots,n-1$ *and for all* $x \in G$, *then an element* $e \in G$ *is a neutral element of* (G,f) *and an* n-*quasigroup* (G,f) *is an* n-*loop*.

Proposition 6. *An* AS-n-*groupoid* (G,f) *is an* n-*loop iff there exists an element* $e \in G$ *such that the equation (5) holds for some fixed* $0 \le i < n$ *and for all* $x \in G$.

Proof. Let (5) holds for some fixed $i = j_0 \in \{0,1,2,\dots,n-1\}$. Since $(i,n+1,i+1) \in A_{n+1}$ for every $i \in N_{n-1}$, then (5) implies $f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$ for every $i = 0,1,2,\dots,j_0$. Similarly the permutation $(i+1,i+2,n+1) \in A_{n+1}$ implies that (5) holds for all $i = j_0, j_0+1, j_0+2,\dots,n-1$. Hence an element $e \in G$ is an identity of (G,f) and (G,f) is an n-ary loop.

The second part of this proposition is obvious.

One can prove (see for example [4] or [7]) that in a Menger n-quasigroup (G,f) there exists *a special element* $e \in G$ such that

$$f(\overset{(n-1)}{e}, x) = f(\overset{(n-1)}{x}, e) = x \tag{6}$$

for all $x \in G$.

From Proposition 6 follows that in an AS-n-quasigroup this element is a neutral element, i.e. that *a Menger* AS-n-*quasigroup is a Menger* n-*loop. This* n-*loop has only one neutral element* [2]. A ternary loop is trivial, i.e. has only one element.

With a Menger n-quasigroup (G,f) is connected so-called *a diagonal group*, i.e. a group $(G,\cdot)$ defined by the formula

$$x\cdot y = f(x,\overset{(n-1)}{y}) .$$

This group *is a retract if* (G,f) (see [2] and [4]). Hence a Menger n-group has the form $f(x_1^n) = x_1x_2x_3 \ldots x_n$, where $(G,\cdot)$ is its diagonal group (see [4]). If this n-group is also alternating symmetric, then its diagonal group is Boolean by Theorem 2. But a Menger n-quasigroup has only one element satisfying (6) (see [2] and [4]), hence this element is an identity of (G,f) by Proposition 6.

Since a Boolean group has the order 2^t, $t \in N$, and for every $t \in N$ there exists such group, then we have the following proposition.

Proposition 7. *If* n *is odd, then a Menger* AS-n-*-groupoid is trivial. If* n *is even, then there exists a nontrivial finite Menger* AS-n-*group of order* q *iff* $q = 2^t$, $t \in N$. *This* n-*group has the form* $f(x_1^n) = x_1x_2x_3 \ldots x_n$, *where* $(G,\cdot)$ *is a Boolean group.*

Finally we consider AS-n-groupoids with central elements.

We say that an element $a \in G$ is *the* i-*th central element* of (G,f) iff

$$f(a,x_2^n) = f(x_2^i,a,x_{i+1}^n)$$

holds for all $x_1^n \in G$. The set of all i-th central elements of (G,f) is called *the* i-*th center of* (G,f) and is denoted by $Z_i(G,f)$.

Since $(i-2,i-1,i)$ and $(i,i+1,i+2)$ are even permutations for every $3 \le i \le n-2$, then

$$f(x_2^{i-2},a,x_{i-1}^n) = f(x_2^i,a,x_{i+1}^n) = f(x_2^{i+2},a,x_{i+3}^n),$$

which shows that $Z_i(G,f) = Z_{i+2}(G,f)$ for every $3 \leq i \leq n-2$. But $Z_1(G,f) = G$, hence $Z_j(G,f) = G$ for every odd $j \in N_n$, and $Z_2(G,f) = Z_k(G,f)$ for every even $k \in N_n$. Moreover, if $G = Z_i(G,f)$ for some $i \in N_n$, then for $\sigma = (1,2,3,\dots,i)$ we have $f = f^\sigma$. Hence we obtain the next proposition.

Proposition 8. *Let* (G,f) *be an* AS-n-*quasigroup. Then* $Z_i(G,f) = G$ *for every odd* $i \in N_n$ *and* $Z_j(G,f) = Z_2(G,f)$ *for every even* $j \in N_n$. *Moreover, if* $Z_k(G,f) = G$ *for some even* $k \in N_n$, *then* (G,f) *is a totally symmetric* n-*quasigroup.*

REFERENCES

[1] В.Д. Белоусов, n-арные квазигруппы, Штиинца, Нишинев, 1972.

[2] В.Д. Белоусов, М.Д. Сандик, n-арные квазигруппы и лупы, Сиб. мат. журнал, 7 (1966), 31 - 54.

[3] W.A. Dudek, *Remarks on n-groups, Demonstratio Math.*, *13* (1980), 165 - 181.

[4] W.A. Dudek, *On some group-like Menger n-groupoids, Proc. of the II Symp. n-ary structures, Varna '83 (in print)*.

[5] W.A. Dudek, Z. Stojaković, *Permutable n-groupoids, Zbornik radova prir.-mat. fak. Univ. u Novom Sadu, Ser. mat.*, 14 (1984), No. 2, 155 - 166.

[6] D.G. Hoffman, *On the spectrum of n-quasigroups with given conjugate invariant subgroup, J. Comb. Theory, ser. A*, 35 (1983), 98 - 99.

[7] H.L. Skala, *Grouplike Menger algebras, Fund. Math.*, 79 (1973), 199 - 207.

[8] Z. Stojaković, *Cyclic n-quasigroups, Zbornik radova Prir.-mat. fak. Univ. u Novom Sadu, Ser. mat.*, 12 (1982), 407 - 415.

[9] Z. Stojaković, *Alternating symmetric n-quasigroups, Zbornik radova Prir.-mat. fak. Univ. u Novom Sadu, Ser. mat.*, 13 (1983), 259 - 272.

REZIME

PRIMEDBE O ALTERNATIVNIM n-KVAZIGRUPAMA

Alternativne simetrične n-kvazigrupe je uveo i opisao Z. Stojaković u [9] kao jednu generalizaciju semisimetričnih kvazigrupa.

U ovoj noti generalisan je jedan deo rezultata Z. Stojakovića i date neke veze izmedju alternativnih simetričnih i totalno simetričnih n-kvazigrupa. Opisani su specijalni elementi alternativnih simetričnih n-kvazigrupa.

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