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BASES OF BOOLEAN FUNCTIONS UNDER CERTAIN COMPOSITIONS

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Abstract.

The notion of completeness of a set of logical functions depends both on the way one is allowed to construct his network from the given set of logical primitives and on logical function the network is defined to realize. In this paper we refer three ways of these constructions and classify all Boolean functions into equivalence classes for each case, then we enumerate classes of bases under corresponding constructions.

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1. Introduction

Consider the following situation arising in the synthesis of switching circuits. We are given certain basic elements called gates which are realizations of certain logical functions (let F denote the set of the logical functions). These gates can be combined into switching circuit by attaching outputs of certain gates to inputs of certain gates so that the resultung circuit has a single output. This can be represented by compositions of the functions from F. We call this construction network. For each network we distinguish inputs and an output. Thus a network can be represented by $f(x_1,\ldots,x_n)$, which defines output $y=f(x_1,\ldots,x_n)$ as a function of the input $x_1,\ldots,x_n$.

In this paper we refer three different ways of the construction of networks which are connected with practical situations in switching circuit designs, and we present all bases for these constructions by dividing them into classes. These results will demonstrate the relation between the conditions of constructions and bases. In the next section we give a short preliminaries for functional completeness problems and maximal sets of classical Post compositions. In Section 3 we treat completeness up to coding (under composition with primitives without delay), in Section 4 completeness under uniform delay compositions (with unit delay primitives), and in Section 5 sequential circuit completeness (with unit delay primitives).

2. Functional completeness

The set of logical functions, i.e. the union of all the functions $\{f \mid E_2^n \to E_2,$ for $E_2=\{0,1\}$ and $n = 0,1,2,\ldots\}$ is denoted by P_2. The operation of superposition (or composition) of functions is defined in the following way (cf. [8,16]):

If f, g are m-ary and n-ary functions from a set $F \subseteq P_2$ then each function obtained from f by permuting and identifying variables and the (m+n-1)-ary function h defined by setting

(1) $h(x_1,\ldots,x_{m+n-1}) = f(g(x_1,\ldots,x_n),x_{n+1},\ldots,x_{m+n-1})$

for all $x_1,\ldots,x_{m+n-1} \in \{0,1\}$ is a **superposition** of functions from F. A subset F of P_2 is said to be **closed** if it contains all superpositions of its members. We will consider situations where several conditions are posed on composition in Sections 3, 4 and 5. For closed sets F and H such that $F \subset H$ (proper inclusion), F is **H-maximal** set if there is no closed set G such that $F \subset G \subset H$. A subset X of H is **complete** in H if H is the least closed set containing X. If the number m of H-maximal sets is finite then a subset of functions in H is complete in H if and only if it is not contained in any one H-maximal set (completeness condition)(cf. [11]). Investigations of completeness and related topics, which are usualy called functional completeness problems are directly related to logical circuit design, and have a wide area of applications in addition to their mathematical importance.

A complete set X in H is called **base of H** if no proper subset of X is complete in H. A set of functions $\{f_1,\dots,f_s\}$ is called **pivotal** in H, if for each i, $1 \le i \le s$, there exists an H-maximal set H_l which does not contain f_i while all the other functions f_j $(j=1,\dots,s, j \neq i)$ are elements of H_l (pivotalness condition). From these definitions follows that a complete pivotal set is a base. In case of no confusion, we say **pivotal incomplete** set simply pivotal. The **rank** of a base (pivotal) is the number of its elements.

We classify the set H of functions into nonempty equivalence classes by using all its maximal sets H_i $(1 \le i \le m)$ as indicated below. Then we can discuss the completeness in H in terms of these classes instead of individual functions: if a set is complete, then by replacing a function in the set by any function in the corresponding equivalence class yields another complete set.

The **characteristic vector** of $f \in H$ is $a_1 \dots a_m$, where $a_i=0$ if $f \in H_i$ and $a_i=1$ otherwise $(1 \le i \le m)$. All functions $f \in H$ with the same characteristic vector form a **class of functions**. For a given set $F \subseteq H$ the **class of F** is the set of classes of functions belonging to F. The conditions of completeness and pivotalness of F can be conveniently checked by using characteristic vectors corresponding to the class of F. All bases (pivotals) with the same class form a **class of bases (pivotals)**. If we have a complete list of characteristic vectors for nonempty classes of functions of a set, we can enumerate all its classes of bases (pivotals) [17].

The notion of completeness depends on the allowed way of constructions of functions and this results different bases.

We give several subsets of P_2 which are useful for our purpose.

$C=\{0,1\}\cup\{x_{i_1}\cdots x_{i_l};\ \{i_1,\dots,i_l\}\subseteq\{1,\dots,n\},\ n\geq 1\}$
(set of all conjunctions).

$D=\{0,1\}\cup\{x_{i_1}\vee\dots\vee x_{i_l};\{i_1,\dots,i_l\}\subseteq\{1,\dots,n\},\ n\geq 1\}$
(set of all disjunctions).

It is easy to see that these two sets are closed but not complete in P_2 under superpositions. We use the notation of functions preserving a relation to describe classical P_2-maximal sets (cf. [15]). An h-ary relation ρ on E_2, $h\geq 1$, is a subset of E_2^h whose elements are written as columns

$$(a_1,\dots,a_h)^T\in\rho \Leftrightarrow (a_{1i},\dots,a_{hi})^T\in\rho \text{ for all } i,\ 1\leq i\leq h,$$
$$\text{where } a_i=(a_{i1},\dots,a_{in}).$$

The relation ρ is written as a matrix whose columns are elements of the relation ρ.

Then set of functions preserving ρ (denoted by Pol ρ) is defined by

$$\text{Pol } \rho=\{f \mid (a_1,\dots,a_h)^T\in\rho \Rightarrow (f(a_1),\dots,f(a_h))^T\in\rho\}.$$

Throughout this paper by x+y and xy we denote x+y(mod 2) and xy(mod 2) respectively. Intersection of sets $X_1,\dots,X_r$ will be denoted by $X_1\dots X_r$.

Theorem 2.1. [15] P_2 has exactly the following 5 maximal sets under superpositions: T_0, T_1, S, L and M, where

$T_0=\text{Pol}(0)=\{f \mid f(0,\dots,0)=0\}$ (set of functions preserving 0).

$T_1=\text{Pol}(1)=\{f \mid f(1,\dots,1)=1\}$ (set of functions preserving 1).

$S=\text{Pol}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\{f \mid f(x_1+1,\dots,x_n+1)\neq f(x_1,\dots,x_n)$
for each $x_i\in\{0,1\}, 1\leq i\leq n\}$ (set of selfdual functions).

$M=\text{Pol}\begin{pmatrix}0&1&0\\0&1&1\end{pmatrix}=\{f \mid x_1\leq y_1\wedge\dots\wedge x_n\leq y_n\Rightarrow f(x_1,\dots,x_n)\leq f(y_1,\dots,y_n)\}$
(set of monotone non-decreasing functions).

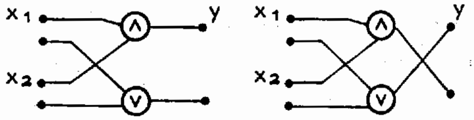
$L=\text{Pol}(\{(a,b,c,d)^T\in E_2^4 \mid a+b=c+b\})$
$=\{f \mid f(x_1,\dots,x_n)=a_0+a_1x_1+\dots+a_nx_n \text{ for some } a_i\in E_2,\ 0\leq i\leq n\}$
(set of linear functions).

The 15 classes of functions and 42 classes of bases of P_2 are well-known [7,4,9]. Hereafter we are interested in the completeness only in P_2, so we don't mention this in the statements.

3. Completeness up to coding

The notion of completeness up to coding is introduced by Freivald in [2]. In this construction every input and output of the outermost network consists of "r-lines" and signals 0 or 1 are feeded to each input or taken out from the output as a length r binary code. While internal network treat these input lines as usual input. So in the internal networks every composition is done according to usual composition. In Fig. 3.1. we show examples of networks of AND and NAND with the coding 0 → 01 and 1 → 10. Note that in this coding negation of the outermost network is realized simply by exchanging the output lines, so if f is realizable then its negation is also realizable in this composition.

Fig. 3.1.



$y = x_1 \wedge x_2$ and $y = \overline{x_1 \wedge x_2}$

Assume a coding

$0 \rightarrow \alpha_{01} \ldots \alpha_{0r}$,

$1 \rightarrow \alpha_{11} \ldots \alpha_{1r}$, where $\alpha_{ij} \in \{0, 1\}$.

We shall say that a network compute $f(x_1, \ldots, x_n)$ with the coding, if , to each argument x_i there is associated the r inputs a_{ij} $(j = 1, \ldots, r)$, the network has r output b_l $(l=1, \ldots, r)$ and operates as follows: for the computation of $f(m_1, \ldots, m_n)$ one feed in α_{m_i} at input line a_{ij} and the network produces as output b_l the results $\beta_l = \alpha_{f(m_1, \ldots, m_n)l}$. We shall say that $F \subseteq P_2$ is complete under a fixed coding if every $f \in P_2$ is computable with the coding by a network on F. We shall say that F is **complete up to coding** (u.t.c-complete), if for every $f \in P_2$, there exists a coding (depending on f) under which f is computable by a network on F.

Theorem 3.1. [2] There exist exactly three u.t.c-maximal sets. They are L, C and D.

Theorem 3.2. There exists exactly 5 classes of functions, 4 classes of bases and 3 classes of pivotals under u.t.c-completeness. The classes are shown in Table 3.1.

Proof. $LD \subseteq C$, $LC \subseteq D$ and $CD \subseteq L$, i.e. $LCD = \{0, 1, x_i \mid 1 \leq i \leq n\}$. We give representative(s) for each class. □

Table 3.1. Classes of functions under u.t.c-completeness.

	L C D	representatives
1.	0 0 0	$\{0, 1, x_i\}$
2.	0 1 1	$a+x_1+\ldots+x_n$, $a=0$ or 1, $n > 1$
3.	1 0 1	$x_1\ldots x_n$, $n > 1$
4.	1 1 0	$x_1 \vee x_2 \ldots \vee x_n$, $n > 1$
5.	1 1 1	$\bar{x}_1\bar{x}_2$.

The classes of bases are : rank 1: (5), rank 2 : (2,3),(2,4),(3,4). The classes of pivotals are: rank 1: (2),(3) and (4).

4. Completeness under uniform delay compositions

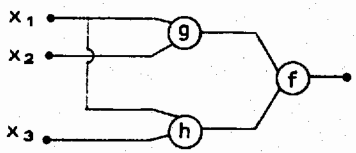
The theory of uniform delay composition was initiated by Kudryavcev [9]. Here each primitive function is assumed to have a unit time delay for its computation, and every composition is to be done so that the delays caused at each part of the composition are synchronized. A set $F \subseteq P_2$ is complete under uniform delay compositions if one can generate every function in some dealy (which depends on the realized function) by a network on F. Furthermore in this composition it is assumed that

(1) all initial input signals are given only once and simultaneously,

(2) no feedback connections are allowed in compositions.

For example, the network in Fig. 4.1 is synchronized, but one in Fig. 5.1 is not synchronized and have a feedback connection.

Fig. 4.1.



An example of uniform delay composition $y(x_1,x_2,x_3)=f(g(x_1,x_2),h(x_1,x_3))$.

The following theorem is proved in [10], but explicit statement in this form is due to Nozaki [11].

Theorem 4.1. [9] There are exactly 8 maximal sets under uniform delay composition and they are T_0, T_1, L, S, M, M', X and K, where

$M' = \{f \mid f(x_1,\ldots,x_n) \geq f(y_1,\ldots,y_n)$ if $x_i \leq y_i$ for all $i\}$

(monotone decreasing functions),

$X = \{ f \mid f(x,\dots,x)=\bar{x} \}$,

$K = \{ f \mid f(0,\dots,0)=f(1,\dots,1) \}$.

Ibuki [5] defined slightly different compositions independently, and gave all 7 maximal set, which coincide with above sets except K. He determined 18 clases, which are closely connected with our classes.

Theorem 4.2. There are 18 classes under uniform delay composition and they coincide with Ibuki's classes.

Proof. Characteristic vector for these classes has 8 coordinates, which is constructed by adding K coordinate to Ibuki's coordinates. We have $K=T_0\bar{T}_1 \cup \bar{T}_0T_1$, because $f \in K \Leftrightarrow f(0)=f(1)=0$ or $f(0)=f(1)=1$. Therefore the values for the coodinate k is determined by those for T_0 and T_1. □

In Table 4.1 we give classes and representatives.

Table 4.1. Classes of functions under uniform delay compositions.

	T_0	T_1	S	L	M	M'	X	K	representative
1.	1	1	1	1	1	1	0	1	$x_1\bar{x}_2x_3 \vee \bar{x}_1\bar{x}_2\bar{x}_3$
2.	1	1	1	1	1	0	0	1	$\bar{x}_1\bar{x}_2$
3.	1	1	0	1	1	1	0	1	$x_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_3x_1$
4.	1	0	1	1	1	1	1	0	$x_1 \vee \bar{x}_2$
5.	0	1	1	1	1	1	1	0	$1x_1\bar{x}_2$
6.	0	0	1	1	1	1	1	1	$x_1x_2x_3+x_1\bar{x}_2\bar{x}_3$
7.	1	1	0	1	1	0	0	1	$\bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_3\bar{x}_1$
8.	1	1	0	0	1	1	0	1	$x_1+\bar{x}_2+x_3$
9.	1	0	1	0	1	1	1	0	$x_1+\bar{x}_2$
10.	0	1	1	0	1	1	1	0	x_1+x_2
11.	0	0	1	1	0	1	1	1	x_1x_2
12.	0	0	0	1	1	1	1	1	$x_1x_2 \vee x_2\bar{x}_3 \vee \bar{x}_3x_1$
13.	1	1	0	0	1	0	0	1	$\bar{x}_1$
14.	0	0	0	1	0	1	1	1	$x_1x_2 \vee x_2x_3 \vee x_3x_1$
15.	0	0	0	0	1	1	1	1	$x_1+x_2+x_3$
16.	1	0	1	0	0	0	1	0	1
17.	0	1	1	0	0	0	1	0	0
18.	0	0	0	0	0	1	1	1	x_1

Although the classes of uniform delay composition coincide with those under Ibuki's composition. the bases and pivotals are different due to the coordiante K.

Theorem 4.3. There are exactly 118 classes of bases and 115 classes of pivotals under uniform delay compositions. They are given below.

Table 4.1. Classes of bases under uniform delay compositions.

rank 1: none;
rank 2: (1) × {4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 17, 18},
(2) × {4, 5, 6, 9, 10, 11, 12, 14, 15, 18},
(3) × {4, 5, 6, 9, 10, 11, 16, 17},
(4) × {7, 8, 13}, (5) × {7, 8, 13}, (6) × {7, 8, 13},
(7) × {9, 10, 11}, (8, 11), (11, 13);
rank 3: (2, 8) × {16, 17}, (4, 5) × {6, 11, 12, 14, 15, 18},
(4, 6) × {10, 17}, (4, 10) × {11, 12, 14, 15, 18},
(4, 11, 17), (4, 12, 17), (4, 14, 17), (4, 15, 17),
(4, 17, 18), (5, 6) × {9, 16},
(5, 9) × {11, 12, 14, 15, 18}, (5, 11, 16), (5, 12, 16),
(5, 14, 16), (5, 15, 16), (5, 16, 18), (6, 9) × {10, 17},
(6, 10, 16), (6, 16, 17), (7, 8) × {16, 17},
(7, 12) × {16, 17}, (7, 14) × {16, 17}, (7, 15) × {16, 17},
(7, 16, 18), (7, 17, 18), (8, 9) × {12, 14}, (8, 10) × {12, 14},
(8, 12) × {16, 17}, (8, 14) × {16, 17}, (9, 10) × {11, 12, 14},
(9, 11, 17), (9, 12) × {13, 17}, (9, 13, 14), (9, 14, 17), (10, 11, 16),
(10, 12) × {13, 16}, (10, 13, 14), (10, 14, 16), (12, 13) × {16, 17},
(12, 16, 17), (13, 14) × {16, 17};
rank 4: (11, 15, 16, 17), (14, 15, 16, 17).

Table 4.2. Classes of pivotals under uniform delay compositions.

rank 1: (1) - (17);
rank 2: 2 × {3, 8, 16, 17}, 3 × {12, 14, 15, 18},
4 × {5, 6, 10, 11, 12, 14, 15, 17, 18}, 5 × {6, 9, 11, 12, 14, 15, 16, 18},
6 × {9, 10, 16, 17}, 7 × {8, 12, 14, 15, 16, 17, 18},

8 × { 9, 10, 12, 14, 15, 16, 17, 18 }, 9 × { 10, 11, 12, 13, 14, 15, 17, 18 },
10 × { 11, 12, 13, 14, 15, 16, 18 }, 11 × { 12, 15, 16, 17 },
12 × { 13, 16, 17 }, 13 × { 14, 15, 16, 17, 18 },
14 × { 15, 16, 17 }, 15 × { 16, 17 }, 16 × { 17, 18 }, (17, 18)}.

rank 3: (9, 10) × { 15, 18 }, (9, 15, 17), (9, 17, 18), (10, 15, 16),
(10, 16, 18), (11, 15) × { 16, 17 }, (11, 16, 17), (13, 15) × { 16, 17 },
(13, 16, 18), (13, 17, 18), ((14, 15) × { 16, 17 }, (14, 16, 17),
(15, 16, 17), (16, 17, 18).

Note that under Ibuki's composition we have 93 bases (2≦rank≦4) [4]. We also note that no Sheffer class exists in our case as well as in Ibuki's one.

5. Sequential circuit completeness

Compositions allowing loops by using unit delay primitives and the notion of S-completeness are introduced by Nozaki [13] (S for sequential circuit). In Fig. 5.1 we show an example of our network. Note that we don't require uniform delay any more. We borrow definitions from [12]. Assume that in our network there are m primitives whose output is denoted by u_i (1≦i≦m) and n inputs denoted by $x_1, \ldots, x_n$. The output of the first primitive u_1 is assumed to be the output of the network. Output of the primitive i after unit delay is denoted by u^*_i and is expressed by

$$u^*_1=D_1(u_1, \ldots, u_m, x_1, \ldots, x_n),$$
$$\ldots\ldots$$
$$u^*_m=D_m(u_1, \ldots, u_m, x_1, \ldots, x_n).$$

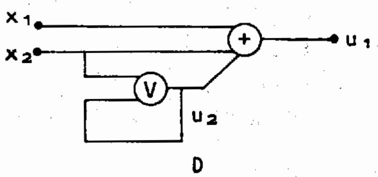


Fig. 5.1. An example of sequential circuit $D(x_1,x_2)=x_1+x_2+1$ with unit delay.

For example, in Fig. 5.1 we have u^*_1=ADD(x_1, x_2, u_2) and u^*_2=OR(x_2, u_2).

Let $Q=\{0, 1\}^m$ and $Y=\{0, 1\}^n$ correspond to the sets of states of the primitives and inputs of the network respectively. Then the network is described by a function

$$D:\ Q \times Y \rightarrow Q$$

and the first element of Q is the output of the network. For example, in Fig. 5.1 D((1,0),(1,0))=(1,0). The state transition of D under feeding x(1)...x(t) to an initial state s(1) is determined successively by s(2)=D(s(1),x(1)), s(3)=D(s(2),x(2)) ..... s(t+1)=D(s(t),x(t)).

The last state s(t+1) is denoted by $D^*(s(1),\alpha)$ and called **final state** corresponding to the input sequence α = x(1),...,x(t), and the first component of s(t+1) is **final output** denoted by $D^{\circ}(s(1),\alpha)$. The notion of realization of function f by a network D is defied as follows.

(1) There exists an initial state s(1) such that
for any input sequence x(1),....,x(T),
let $y(t+1)=D^{\circ}(s(1),x(1)...x(t))$. Then
$y(t+d)=f(x(t))$ for $1 \le t \le T-d+1$.
Any such state s(1) is called a **good state**.

(2) For any state s there is an input sequence α such that $D^*(s,\alpha)$ is a good state. Such α is called an initialize sequence.

In Fig. 5.1 D realizes x+y+1 with initialize sequence (0,1) or (1,0) with delay 1.

We denote the set of all functions realizable with some delay by a network on F by $[F]_s$. F is called s-complete if $[F]_s = P_2$. We denote by $[F]_c$ the set of all functions realizable by a network which is constructed by combinatorial switching circuit on F (i.e. by loop-free connections). If $[F]_c = P_2$ then F is called c-complete. The notion of c-completeness is defined under composition which allow nonuniform delay and coincides with that of Ibuki's construction [5] of which we refered in Section 4. From the definitions we have $[F]_c \subseteq [F]_s$, hence if F is c-complete then it is s-complete. Let us introduce two subsets of P_2.

$$N_0=\mathrm{Pol}\begin{pmatrix}0&1&0\\1&0&0\end{pmatrix}=\{f \mid f(x_1,\dots,x_n)=f(y_1,\dots,y_n)=1 \Rightarrow x_i=y_i=1 \text{ for some } i,\ 1\le i\le n\},$$

$$N_1=\mathrm{Pol}\begin{pmatrix}0&1&1\\1&0&1\end{pmatrix}=\{f \mid f(x_1,\dots,x_n)=f(y_1,\dots,y_n)=0 \Rightarrow x_i=y_i=0 \text{ for some } i,\ 1\le i\le n\}.$$

Theorem 5.1. [12] There are exactly 6 maximal sets under s-completeness.

They are N_0, N_1, S, L, M and M'.

Theorem 5.2. There are exactly 15 classes of functions under s-completeness. They are indicatd in Table 5.1.

Proof. To have these classifications we used classes with respect to T_0, T_1, S, L, M and M' in [5], and $N_0 \subset T_0$ and $N_1 \subset T_1$. We list several useful formulas for the classification. $S \subseteq N_0N_1 \cup \bar{N}_0\bar{N}_1$, $N_0N_1 \subseteq S$, $MM'=\{0, 1\}$, $ML=\{0, 1, x_i\}$, $M'L=\{0, 1, x_i+1\}$, $SL=\{1+x_1+\ldots+x_{2m+1}\}$, $LN_0=\{0, x_i\}$, $LN_1=\{1, x_i\}$, $SN_0 \subseteq M$ and $SM \subseteq N_0$. □

Table 5.1. Classes of functions under s-completeness.

	N_0	N_1	S	L	M	M'	representative
1.	1	1	1	1	1	1	$x_1\bar{x}_2 \vee \bar{x}_1x_2 \vee \bar{x}_3$
2.	1	1	1	1	1	0	$\bar{x}_1\bar{x}_2$
3.	1	1	1	0	1	1	x_1+x_2
4.	1	1	0	1	1	1	$x_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_3x_1$
5.	1	0	1	1	1	1	$x_1 \vee \bar{x}_2$
6.	0	1	1	1	1	1	$x_1\bar{x}_2$
7.	1	1	0	1	1	0	$\bar{x}_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_3\bar{x}_1$
8.	1	1	0	0	1	1	$x_1+x_2+\bar{x}_3$
9.	1	0	1	1	0	1	$x_1 \vee x_2$
10.	0	1	1	1	0	1	x_1x_2
11.	1	1	0	0	1	0	$\bar{x}_1$
12.	1	0	1	0	0	0	1
13.	0	1	1	0	0	0	0
14.	0	0	0	1	0	1	$x_1x_2 \vee x_2x_3 \vee x_3x_1$
15.	0	0	0	0	0	1	x_1

Theorem 5.3. There are exactly 50 classes of bases and 40 classes of pivotals under s-completeness. They are indicated in Table 5.2 and 5.3 respectively.

Table 5.2. Classes of bases under s-completeness.

rank 1: (1);

rank 2: (2) × {3, 4, 5, 6, 8, 9, 10, 14, 15}, (3) × {4, 5, 6, 7, 9, 10, 14},
(4) × {5, 8, 9, 10, 12, 13}, (5) × {6, 7, 8, 10, 11, 13},
(6) × {7, 8, 9, 11, 12}, (7) × {9, 10}, (8) × {9, 10},
(9, 11), (10, 11);
rank 3: (7, 8) × {12, 13}, (7, 12) × {14, 15}, (7, 13) × {14, 15},
(8, 12, 14), (8, 13, 14), (11, 12, 14), (11, 13, 14).

Table 5.3. Classes of pivotals under s-completeness.

rank 1: (2) - (15);
rank 2: (7) × {8, 12, 13, 14, 15}, (8) × {12, 13, 14}, (9) × {10, 13},
(10, 12), (11) × {12, 13, 14, 15}, 12 × {13, 14, 15}, 13 × {14, 15};
rank 3: (11, 12, 15), (11, 13, 15), (12, 13) × (14, 15).

6. Concluding remarks

The completeness problem under fixed coding 0 → 01 and 1 → 10 was solved by Ibuki [5]. He gave all 6 maximal sets: N_0, N_1, S, L, C and D. He determined 12 classes and 28 bases (1≦rank≦3). A notion very similar to c-completeness was proposed by Inagaki [6]. He gave exacly 6 maximal sests: T_0, T_1, S, L, M and M'. He determined 18 classes and 82 bases (1≦rank≦4) and noted that in this case, in contrast to Kudryavtsev and Ibuki cases, there exists a Scheffer class. Another modification is algebra ϕ^0 proposed by Cejtlin [1]. Classifications and base consideration was done for this case by Tosić [18]. Several other modifications of propositional algebras are considered in [3].

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REZIME

BAZE ZA NEKE MODIFIKACIJE ISKAZNE ALGEBRE

Pojam kompletnosti nekog skupa logičkih funkcija zavisi od načina na koji se logičke funkcije mogu povezati u prekidačku šemu. U ovom radu tri različita načina su razmatrana i za svaki od njih odgovarajuce baze su odredjene i klasifikovane.

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