ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 15,2 (1985)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 15, 2 (1985)

ENUMERATION OF BASES OF SEMI-DEGENERATE, LINEAR AND SELF-DUAL FUNCTIONS OF PRIME--VALUED LOGICS

Ivan Stojmenović

Institute of Mathematics University of Novi Sad,
Dr. Ilije Djuričića 4, 21000 Novi Sad, Yugoslavia

ABSTRACT

The paper determines the classes of functions and classes of bases and pivotal sets for the set S_p of selfdual functions of p-valued logic (p-prime number). For this set and the sets L_p (set of linear functions of p-valued logic) and T (set of semi-degenerate or Slupecki functions of three--valued logic) there are determined the numbers of n-ary functions and symmetric n-ary functions for each class of functions, the numbers of bases, pivotal sets, symmetric bases and symmetric pivotal sets of each rank, consisting of n-ary functions or consisting of functions depending on n variables at most.

1. INTRODUCTION

The set of k-valued logical functions, i.e. the union of all the functions $\{f|E_k^n \to E_k\}$ for $E_k = \{0,1,\ldots,k-1\}$ and $n = 0,1,2,\ldots$ is denoted by P_k.

An n-ary function $f(x_1,\ldots,x_n)$ is said to be symmetric iff the following equality is valid:

AMS Mathematics Subject Classification(1980): Primary 03B50; Secondary 05A15, 03G20.

Key words and phrases: Three-valued logic algebra, maximal clones, bases, pivotal sets, enumeration, classes of functions, symmetric functions, self-dual functions, linear functions, semi--degenerate functions.

$$f(x_1,\dots,x_n) = f(y_1,\dots,y_n),$$

where $(y_1,\dots,y_n)$ is an arbitrary permutation of $(x_1,\dots,x_n)$. A subset F of P_k is said to be closed if it contains all superpositions of its members (cf. [2]). A closed set $F \subset P_k$ is P_k-maximal set if there is no closed set G such that $F \subset G \subset P_k$.

A clone F is a subset of P_k containing the projection functions x_i and closed with respect so superposition.

A subset of functions F is complete in P_k if P_k is the least closed set containing F. A complete set X in P_k is called base in P_k if no proper subset of X is complete in P_k. A subset of functions is complete in P_k iff it is not contained in any one P_k-maximal set ([2]).

A set of functions $\{f_1,\dots,f_s\}$ is called pivotal in P_k iff for each i $(1 \le i \le s)$ there exists a P_k-maximal set H_ℓ which does not contain f_i while all the other functions f_j $(j = 1,\dots,s,\ j \neq i)$ are elements of H_ℓ. From these definitions it follows that a complete pivotal set is a base. The rank of a base (pivotal set) is the number of its elements.

S-base (S-pivotal set) is a base (pivotal set) which contains only symmetric functions.

The characteristic vector of $f \in P_k$ is $a_1 \dots a_m$, where $a_i = 0$ if $f \in H_i$ and $a_i = 1$ otherwise $(1 \le i \le m)$. All functions $f \in P_k$ with the same characteristic vector form a class of functions. Classes of functions for each base (pivotal set) determine the class of given base (pivotal set).

Recently, Lau ([4]) and others give a complete list of submaximal clones of the maximal clones of P_3.

In [3] classes of functions and classes of bases of P_2 are investigated. In [12] the number of n-ary symmetric functions for each of 15 classes of functions of P_2 is determined through which general formulas for the numbers of S-bases are given. There are 406 classes of functions ([6,8]) and 6239721 classes of bases ([8]) of P_3.

In [10] it is proved that 394 classes of functions of

P_3 contain symmetric functions and 12 classes do not contain a symmetric function. Also, in [10] there are determined the number of classes of S-bases and S-pivotal incomplete sets of each rank.

Miyakawa [7] classified three types if maximal clones of P : T (semi-degenerate of Slupecki functions), L (linear functions) and S (self-dual functions), and he enumerated their classes of bases and pivotal incomplete sets. The number of classes of pivotal incomplete sets for the set T given in [7] is incorrect. In this paper we shall present the correct result. Also, in this paper, we shall classity the set S_p of self-dual functions in P_p. In [9] there is given the classification of the set L_p of linear functions in P_p (p-prime number).

Let 0^s denote $\underbrace{00\ldots0}_{s}$ and 1^s denote $\underbrace{11\ldots1}_{s}$. The intersection of sets $X_1,\ldots,X_s$ will be denoted by $X_1,\ldots,X_s$. For $X \subset H$, by $\bar{X}$ we denote the set $H \setminus X$.

The sets T, L_p and S_p are defined in the following way:

$$L_p = \{f \mid f = a_0 + \sum_{i=1}^{n} a_i x_i \pmod p),\ a_i \in \{0,1,\ldots,p-1\},$$
$$0 \le i \le n,\ n \ge 0\},$$

$$S_p = \{f \mid f(x_1+1,\ldots,x_n+1) = f(x_1,\ldots,x_n) + 1 \pmod p)\},$$

$$T = D \cup [P_3^{(1)}]$$

where

$$D = \{f \in P_3 \mid \text{ range of } f \neq \{0,1,2\}\}.$$

$P_3^{(1)}$ is the set of unary functions of P_3. [F] denotes the clone generated from F.

Let, $k/X/(n)$ and $k_s/x/(n)$ denote the numbers of n-ary and n-ary symmetric functions of the set X, respectively. For a clone H, we define the numbers $t_n(j)$, $t_{s,n}(j)$, $t_{\le n}(j)$, $t_{s,\le n}(j)$,

$B_r(n)$, $B_{s,r}(n)$, $B_r(\le n)$, $B_{s,r}(\le n)$, $P_r(n)$, $P_{s,r}(n)$, $P_r(\le n)$ and $P_{s,r}(\le n)$, where $t_n(j)$ and $t_{s,n}(j)$ denote the numbers of n-ary and the symmetric n-ary functions of the class j, $t_{\le n}(j)$ and $t_{s,\ n}(j)$ denote the numbers of functions and symmetric functions which depend on n variables at most. The remaining eight numbers denote the number of bases, and the symmetric bases consisting of n-ary functions and functions depending on n variables at most of rank r, and analogous numbers for pivotal sets.

In this paper, for the sets T, L_p and S_p, the above twelve numbers are obtained.

2. S_p (SELF-DUAL FUNCTIONS)

Theorem 1. ([4]) *S has exactly the following two maximal clones:*

$$S_L = S_pL_p \textit{ and } S_0 = \{f \mid f \in S_p \textit{ and } f(0,\dots,0)=0\}.$$

Theorem 2. *The P_p-maximal clone S_p is divided into four classes:*

class	S_L	S_0
1	1	1
2	1	0
3	0	1
4	0	0

Proof. The function $f(x) = x$ is in the set S_LS_0 and the function $f(x) = x+1 \pmod p$ is in the set $S_L\bar{S}_0$. We define the functions f and g in the following way:

$$f(x,x+1 \pmod p) = x+1 \pmod p,$$
$$f(x,y) = x \text{ for } y \neq x+1 \pmod p,$$
$$g(x,y) = f(x,y) + 1 \pmod p.$$

The function f is in the set $\bar{S}_LS_0$ and the function g is in the set $\bar{S}_L\bar{S}_0$.

Now, we can easily prove the following theorem.

Theorem 3. *S has exactly the following 2 classes of bases and 2 classes of pivotal incomplete sets:*

bases: {1} (rank = 1), {2,3} (rank = 2).
pivotals: {2},{3} (rank = 1).

It is easily proved that $k/S_p/(n) = p^{p^{n-1}}$. *Also, it obviously follows that* $k/S_0/(n) = p_n^{p^{n-1}-1}$. *From* $f \in S_1 \leftrightarrow f(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i \pmod p$ & $\sum_{i=1}^{n} a_i = 1 \pmod p$, *it follows that* $k/S_L/(n) = p \cdot p^{n-1} = p^n$.

Theorem 4.
$$t_n(4) = k/S_L S_0/(n) = p^{n-1};$$
$$t_n(3) = k/S_L \bar{S}_0/(n) = (p-1)p^{n-1};$$
$$t_n(2) = k/\bar{S}_L S_0/(n) = p^{p^{n-1}-1} - p^{n-1};$$
$$t_n(1) = k/\bar{S}_L \bar{S}_0/(n) = (p-1)p^{p^{n-1}-1} - (p-1)p^{n-1}.$$

Proof. Follows immediately from the definitions of the sets S_L and S_0.

Theorem 5.
$$k_s/S_p/(n) = \begin{cases} 0 & \textit{for } n = 0 \pmod p \\ \frac{1}{p}\binom{n+p-1}{p-1} & \textit{for } n \neq 0 \pmod p. \end{cases}$$

Proof. From the definition of symmetric functions it follows that the value of an n-ary symmetric function is equal for all vectors consisting of the same number of 0, the same number of 1,..., the same number of p-1. Hence, we can write

$$f[m_0, m_1, \ldots, m_{p-1}] = f(\underbrace{0,\ldots,0}_{m_0}, \underbrace{1,\ldots,1}_{m_1}, \ldots, \underbrace{p-1,\ldots,p-1}_{m_{p-1}}),$$

where $m_0 + m_1 + \ldots + m_{p-1} = n$. The number of p-ary vectors $(m_0, \ldots, m_{p-1})$, $m_0 + \ldots + m_{p-1} = n$ is $\binom{n+p-1}{p-1}$. We shall consider two cases:

1. $n = pm$. Let $f[m, \ldots, m] = a$. From $f[m_0, \ldots, m_{p-1}] = a$ follows $f[m_1, m_2, \ldots, m_{p-1}, m_0] = a+1 \pmod p$, because the function f is a selfdual function. Hence, from $f[m, \ldots, m] = a$, we obtain $f[m, \ldots, m] = a+1$, and this is a contradiction.

2. $n \neq 0 \pmod p$. From $f[m_0, \ldots, m_{p-1}] = a$ follows $f[m_k, m_{k+1}, \ldots, m_{p-1}, m_0, \ldots, m_{k-1}] = a+k$ $(0 \leq k \leq p-1)$. Therefore, function f is determined by values of $\frac{1}{p}\binom{n+p-1}{p-1}$ vectors.

Theorem 6.

$$t_{s,n}(4) = t_{s,n}(3) = t_{s,n}(2) = t_{s,n}(1) = k_s/S_L S_0/(n) =$$

$$= k_s/S_L \bar{S}_0/(n) = k_s/\bar{S}_L S_0/(n) = k_s/\bar{S}_L \bar{S}_0/(n) = 0$$

for $n = 0 \pmod p$

$$t_{s,n}(4) = 1, \quad t_{s,n}(3) = p-1, \quad t_{s,n}(2) = p^{\frac{1}{p}\binom{n+p-1}{p-1}-1} - 1$$

$$t_{s,n}(1) = (p-1)p^{\frac{1}{p}\binom{n+p-1}{p-1}-1} - (p-1)$$

for $n = 0 \pmod p$.

Proof. There are exactly p^2 n-ary symmetric linear functions of P_p. These are

$$a_0 + a_1 \sum_{i=1}^{n} x_i, \quad a_0, a_1 \in \{0, 1, \ldots, p-1\}.$$

A symmetric linear function is self-dual if $na_1 \equiv 1 \pmod p$. Hence, $k_s/S_L/(n) = p$ and

$$k_s/\bar{S}_L/(n) = p^{\frac{1}{p}\binom{n+p-1}{p-1}} - 3 \quad (n \not\equiv 0 \pmod p).$$

A symmetric linear function is in the set S_0, if $a_0 = 0$. Hence, $k_s/S_LS_0/(n) = 1$ and $k_s/S_L\bar{S}_0/(n) = p-1$ ($n \not\equiv 0 \pmod p$). The remaining results follow from $k_s/\bar{S}_LS_0/(n) = \frac{1}{p}k_s/\bar{S}_L/(n)$ and $k_s/\bar{S}_L\bar{S}_0/ = \frac{p-1}{p}\, k_s/\bar{S}_L/(n)$.

From Theorem 3, we obtain:

Theorem 7.

$B_1(n) = t_n(1)$; $B_2(n) = t_n(2)t_n(3)$;

$P_1(n) = t_n(2) + t_n(3)$;

$B_{s,1}(n) = t_{s,n}(1)$; $B_{s,2}(n) = t_{s,n}(2)t_{s,n}(3)$;

$P_{s,1}(n) = t_{s,n}(2) + t_{s,n}(3)$.

From the above results we can easily determine the numbers $t_{\le n}(j)$, $t_{s,\le n}(j)$, $B_r(\le n)$, $B_{s,r}(\le n)$, $P_r(\le n)$ *and* $P_{s,r}(\le n)$, *also.*

3. L_p (LINEAR FUNCTIONS OF P_p)

We shall consider only nondegenerated functions of the set L_p.

Let us define some closed sets in P_k.

The set of linear functions L_p is defined in the following way:

$$L_0 = a_0 + \sum_{i=1}^{n} a_i x_i \pmod p, \qquad a_0 \in \{0,1,\dots,p-1\},$$

$a_i \in \{1,2,\dots,p-1\}, \quad 1 \le i \le n, \ n \ge 0\}.$

Let $\sum_{i=1}^{n} a_i$.

The set of linear functions in P_k is a P_k-maximal clone, iff k is a prime number ([3]).

The set of self-dual functions S_p is defined as follows:

$$S_p = \{f \mid f(x_1 + 1,\dots,x_n + 1) =$$

$$= f(x_1,\dots,x_n) + 1 \pmod p, \ n = 1,2,\dots\}.$$

$T_p^m = \{f \mid f(m,\dots,m) = m\}$ is the set of functions preserving $m (0 \le m \le p-1)$.

Theorem 8 ([1]). *There are exactly* $p+2$ L_p*-maximal clones (p is a prime number):*

$$L_p^m = L_p T_p^m, \ m = 0,1,\dots,p-1,$$

$$L_p^p = L_p S_p = \{a_0 + \sum_{i=0}^{n} a_i x_i \mid a = 1 \pmod p)\}$$ *-set of linear self-dual functions,*

$$L_p^{(1)} = \{a_0 + a_1 x \quad a_0, a_1 \in \{0,1,\dots,p-1\}\}$$ *- set of unary linear functions.*

Theorem 9. ([9]). *There are exactly* $2p+4$ *classes of functions of the set* L_p:

1: $L_p^0 L_p^1 \dots L_p^{p-1} L_p^p L_p^{(1)}$, *i.e.* $0,\dots,0$ *or* 0^{p+2},

2: $L_p^0 L_p^1 \dots L_p^{p-1} L_p^p \bar{L}_p^{(1)}$, *i.e.* $0^{p+1}1$, k: $\bar{L}_p^0 \bar{L}_p^1 \ldots \bar{L}_p^{k-4} L_p^{k-3} \bar{L}_p^{k-2} \ldots \bar{L}_p^p L_p^{(1)}$, *i.e.*

$1^{k-3} 0 1^{p+3-k} 0$ $(3 \le k \le p+3)$,

m: $\bar{L}_p^0 \bar{L}_p^1 \ldots \bar{L}_p^{m-p-5} L_p^{m-p-4} \bar{L}_p^{m-p-3} \ldots \bar{L}_p^p \bar{L}_p^{(1)}$, *i.e.*

$1^{m-p-4} 0 1^{2p+5-m}$ $(p+4 \le m \le 2p+4)$.

Theorem 10 ([9]). *The number of classes of bases and pivotal incomplete sets for each rank is shown in the following table:*

rank	*number of classes of bases*	*number of classes of pivotal incomplete sets*
1	0	$2p+3$
2	$3\binom{p+1}{2}$	$\binom{p+1}{2}+p+1$
3	$\binom{p+1}{2}$	0
4	0	0

Bases of rank 2 consist of any two functions of classes i *and* j, *where* i *and* j *satisfy the condition*

- $p+4 \le i < j \le 2p+4$, *or the condition*
- $3 \le i \le p+3$, $p+4 \le j \le 2p+4$ *and* $j \ne i+p+1$.

The base of rank 3 consists of a function of class 2 and two functions of classes i *and* j, *where* $3 \le i < j \le p+3$.

Pivotal incomplete sets of rank 2 consist of

- *two functions of classes* i *and* j, $3 \le i < j \le p+3$,

or - *a function of class 2 and a function of class* i, $3 \le i \le p+3$.

Theorem 11. ([9]).

$t_0(k) = 1$ $\quad 3 \le k \le p+2$, $t_0(k) = 0$ *otherwise*,

$t_1(1) = 1$, $\quad t_1(p+3) = p-1$, $\quad t_1(k) = p-2$ *for* $3 \le k \le p+2$,

$t_1(k) = 0$ *otherwise,*

$t_n(2) = ((p-1)^n - (-1)^n)/p, \quad t_n(2p+4) = (p-1)t_n(2),$

$t_n(k) = ((p-1)^{n+1} + (-1)^n)/p$ *for* $p+4 \le k \le 2p+3,$

$t_n(k) = 0$ *otherwise* $(n \ge 2)$.

Theorem 12. ([9]).

$t_{\le 0}(k) = t_0(k), \quad t_{\le 1}(k) = t_0(k) + t_1(k),$

$t_{\le n}(1) = 1, \; t_{\le n}(k) = p-1$ *for* $3 \le k \le p+3,$

$t_{\le n}(2) = ((p-1)^{n+1} - (p-1)^2)/(p-2) - ((-1)^n + 1)/2)/p,$

$t_{\le n}(2p+4) = (p-1)t_{\le n}(2),$

$t_{\le n}(k) = ((p-1)^2((p-1)^n - 1)/(p-2) + (1 + (-1)^n)/2)/p$

for $p+4 \le k \le 2p+3.$

Theorem 13. ([9]).

$$B_2(n) = pt_n(2p+4)t_n(p+4) + \binom{p}{2}t_n^2(p+4) + pt_n(p+3)t_n(p+4)$$
$$+ pt_n(2p+4)t_n(3) + t_n(p+3)t_n(2p+4) +$$
$$+ p^2t_n(3)t_n(p+4)$$

$$B_3(n) = t_n(2)(pt_n(p+3)t_n(3) + \binom{p}{2}t_n^2(3))$$

$$P_1(n) = t_n(2) + t_n(p+3) + t_n(2p+4) + pt_n(3) + pt_n(p+4)$$

$$P_2(n) = t_n(2)(t_n(p+3) + pt_n(3)) + pt_n(p+3)t_n(3) +$$
$$+ \binom{p}{2}t_n(3)$$

$$B_1(n) = B_4(n) = B_5(n) = B_6(n) = \dots = P_3(n) =$$
$$= P_4(n) = \dots = 0.$$

Analogously, we can obtain the numbers $B_r(\le n)$ *and* $P_r(\le n)$.

A linear symmetric function has the form

$$f(x_1,\dots,x_n) = a_0 + b(x_1 + \dots + x_n) \pmod p,\ n \ge 0,$$
$$1 \le b \le p-1.$$

So, the number of n-ary symmetric linear functions of P_p is $p(p-1)$.

The consideration of symmetric linear functions gives the following theorem:

Theorem 14. *The number of* n*-ary linear symmetric functions of each class is presented in the following table:*

class	*n=0*	*n=1*	n≥2, n≠0(mod p)	n≥2, n=0(mod p)	$k_s/X/(\le n)$, n≥2
1	0	1	0	0	1
2	0	0	1	0	$n-1-[\frac{n}{p}]$
3,4,...,p+2	1	p-2	0	0	p-1
p+3	0	p-1	0	0	p-1
p+4,...,2p+3	0	0	p-2	p-1	$(n-1)(p-2)+[\frac{n}{p}]$
2p+4	0	0	p-1	0	$(p-1)(n-1-[\frac{n}{p}])$

This table presents the numbers $k_s/X/(\le n)$ of symmetric linear functions depending on at most n variables, also.

On the basis of the given data, we obtain the following two theorems:

Theorem 15.

$$B_{s,2}(n) = \begin{cases} \frac{1}{2}p^2(p-1)(p-2) & \text{for } n \neq 0 \pmod p,\ n > 1 \\ \frac{(p-1)^3 p}{2} & \text{for } n = 0 \pmod p,\ n > 1 \end{cases}$$

$$B_{s,3}(n) = 0$$

$$P_{s,1}(n) = \begin{cases} p & \text{for } n = 0 \\ p^2-p-1 & \text{for } n = 1 \\ p^2-p & \text{for } n > 1 \end{cases}$$

$$P_{s,2}(n) = \begin{cases} \frac{p(p-1)}{2} & \text{for } n = 0 \\ \frac{1}{2}p^2(p-1)(p-2) & \text{for } n = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

Theorem 16.

$$P_{s,1}(\leq n) = p^2 n + [\tfrac{n}{p}]^{-np+p-n}$$

$$P_{s,2}(\leq n) = (p^2-1)(n-2p-1-[\tfrac{n}{p}]).$$

Analogously, we can obtain the numbers $B_{s,r}(\leq n)$.

4. T (SEMI-DEGENERATE OR SLUPECKI FUNCTIONS)

For a unary function $f \in P_3^{(1)}$, we denote it by $S_{f(0)f(1)f(2)}$; for example, the identity function is denoted by S_{012}.

Theorem 17 ([5]). *T has exactly the following 5 maximal clones*

1 $T_1 = D \cup [\{S_{012}, S_{021}\}]$

2 $T_2 = D \cup [\{S_{012}, S_{102}\}]$

3 $T_3 = D \cup [\{S_{012}, S_{210}\}]$

4 $T_+ = D \cup [\{S_{012}, S_{120}, S_{201}\}]$

5 $T_b = [P_3^{(1)}] \cup T_U$, *where*

$$T_U = \bigcup_{n=1}^{\infty} \{f^{(n)} \in P_3 \mid \exists f_i \in P_3^{(1)} \quad (0 \le i \le n)$$

such that

$$f(x_1,\dots,x_n) = f_0(f_1(x_1) + f_2(x_2) + \dots$$

$$\dots + f_n(x_n) \pmod 2)\}.$$

Theorem 18. ([7]). T *has exactly the following 6 classes of functions*

class	T_1	T_2	T_3	T_+	T_b
1	0	1	1	1	0
2	1	0	1	1	0
3	1	1	0	1	0
4	1	1	1	0	0
5	0	0	0	0	1
6	0	0	0	0	0

Theorem 19. ([7]). T *has exactly the following 6 classes of bases whose rank = 3:*

{1,2,5}, {1,3,5}, {1,4,5}, {2,3,5}, {2,4,5}, {3,4,5}.

Thus,any base of T consists exactly of three elements.

Theorem 20. T *has exactly the following 15 classes of pivotal incomplete sets:*

rank = 1: *each of 5 classes, except the null class,*

rank = 2: {1,2}, {1,3}, {1,4}, {1,5}, {2,3}, {2,4}, {2,5}, {3,4}, {3,5}, {4,5}.

It is easily proved that $k/T/(n) = 3\cdot 2^{3^n} - 3+6n$.
Now, we shall determine the numbers $t_n(k)$, $1 \le k \le 6$.

First we shall prove the following three lemmas.

Lemma 1. *If* $f(x_1,\dots,x_n) = f_0(f_1(x_1) + \dots + f_n(x_n)$ (mod 2)), $f_i \in P_3^{(1)}$, $0 \le 1 \le n$, *then there exist the functions* $f_i' \in P_3^{(1)}$ $(0 \le i \le n)$ *with the following property:*

$$f(x_1,\dots,x_n) = f_0'(f_1'(x_1) + \dots + f_n'(x_n) \pmod 2)$$

and

$$f_1'(0) = \dots = f_n'(0) = 0.$$

Proof. Suppose that $f_i(0) = 1$ for some i $(1 \le i \le n)$. For functions f_0' and f_i', defined by $f_0'(0) = f_0(1)$, $f_0'(1) = f_0(0)$, $f_i'(x_i) = f_i(x_i) + 1 \pmod 2$, the following equality is satisfied

$$f_0'(f_1(x_1) + \dots + f_{i-1}(x_{i-1}) + f_i'(x_i) + f_{i+1}(x+1) + \dots$$

$$\dots + f_n(x_n) \pmod 2) = f_0(f_1(x_1) + \dots + f_n(x_n) \pmod 2)$$

But $f_i'(0) = 0$. This procedure can be applied for each i which satisfies $f_i(0) = 1$. The lemma is proved.

Lemma 2. *If*

$$f_0'(f_1'(x_1) + \dots + f_n'(x_n) \pmod 2) =$$

$$= f_0''(f_1''(x_1) + \dots + f_n''(x_n) \pmod 2)$$

and

$$f_1'(0) = \dots = f_n'(0) = f_1''(0) = \dots = f_n''(0),$$

$$f_0'(0) \ne f_0'(1),$$

then

$$f_0'(0) = f_0''(0), \quad f_0'(1) = f_0''(1)$$

$$f_i'(x) = f_i''(x) \pmod 2 \quad (1 \le i \le n,\ 0 \le x \le 2).$$

Proof. It follows from $f_1'(0) + \dots + f_n'(0) = f_1''(0) + \dots + f_n''(0) = 0$ that $f_0'(0) = f_0''(0)$. Hence, $f_0'(1) = f_0''(1)$, because $\{f'(0),f'(1)\} = \{f''(0),f''(1)\}$. From $f_1'(0) + \ldots + f_{i-1}'(0) + f_i'(x) + f_{i+1}'(0) + \ldots + f_n'(0) = f_i'(x) \pmod 2$ and $f_1''(0) + \ldots + f_{i-1}''(0) + f_i''(x) + f_{i+1}''(0) + \ldots + f_n''(0) = f_i''(x) \pmod 2$, we obtain $f_0'(f_1'(x)) = f_0''(f_1''(x))$. Therefore, $f_i'(x) = f_i''(x) \pmod 2$. The lemma is proved.

Lemma 3.

$$k/T_U/(n) = 6\cdot 4^n - 3 \quad (n \geq 0).$$

Proof. It follows from the above two lemmas that each of the functions f_i $(1 \leq i \leq n)$ is equal to one of the functions $S_{000}, S_{001}, S_{010}, S_{011}$. The values of $f_0(0)$ and $f_0(1)$ are equal to one of the following pairs: (0,1), (1,0), (0,2), (2,0), (1,2), (2,1). The constant functions 0, 1 and 2 are enumerated twice. The lemma is proved.

Theorem 21.

$$t_n(1) = t_n(2) = t_n(3) = n; \qquad t_n(4) = 2n;$$

$$t_n(5) = 3\cdot 2^{3^n} - 6\cdot 4^n; \quad t_n(6) = 6\cdot 4^n - 3 + n.$$

Proof. For classes 1, 2, 3 and 4, the proof is obvious. Equality for $t_n(6)$ follows immediately from Lemma 3. The function S_{012} is of class 6, also. $t_n(5)$ can be determined by using the equality $k/T/(n) = t_n(1) + t_n(2) + t_n(3) + t_n(4) + t_n(5) + t_n(6)$.

From the above two theorems, we obtain:

Theorem 22.

$$B_3(n) = 9n^2(3\cdot 2^{3^n} - 6\cdot 4^n); \quad B_k(n) = 0 \textit{ otherwise.}$$

$$P_1(n) = 3\cdot 2^{3^n} - 6\cdot 4^n + 5n,$$

$$P_2(n) = (3\cdot 2^{3^n} - 6\cdot 4^n)\, 5n + 9n^2,$$

$P_k^n = 0$ *otherwise.*

Theorem 23. ([11]).

$$k_s/T/(n) = \begin{cases} 27 & \textit{for} \quad n = 1 \\ 3\cdot 2^{\binom{n+2}{2}} - 3 & \textit{for } n > 1. \end{cases}$$

Theorem 24.

$$t_{s,n}(1) = t_{s,n}(2) = t_{s,n}(3) = \begin{cases} 1 \textit{ for } n = 1 \\ 0 \textit{ for } n \neq 1 \end{cases},$$

$$t_{s,n}(4) = \begin{cases} 2 \textit{ for } n = 1 \\ 0 \textit{ for } n \neq 1 \end{cases},$$

$$t_{s,n}(5) = \begin{cases} 0 \textit{ for } n \leq 1 \\ 3\cdot 2^{\binom{n+2}{2}} - 24 \quad n > 1 \end{cases}$$

$$t_{s,n}(6) = \begin{cases} 3 \textit{ for } n = 0 \\ 22 \textit{ for } n = 1 \\ 21 \textit{ for } n > 1 \end{cases}.$$

Proof. The functions of the set $P_3^{(1)}$ are symmetric. Symmetric degenerate functions are only constants 0, 1, 2 ([11]). Hence, symmetric functions of the classes 1, 2, 3 and 4 are only unary functions. The numbers $t_{s,n}(5)$ and $t_{s,n}(6)$ are determined by using the following lemma.

Lemma 4. *If function*

$$f(x_1,\dots,x_n) = f_0(f_1(x_1) + \dots + f_n(x_n) \pmod 2)$$

is a symmetric function, $f_0(0) \neq f_0(1)$ *and* $f_1(0) = f_2(0) = \dots = f_n(0)$, *then* $f_1(x) = f_2(x) = \dots = f_n(x) \pmod 2$.

Proof. Suppose that $f_1(a) \neq f_2(a) \pmod 2$ for some a $(1 \leq a \leq 2)$. From this we conclude $f(a,0,\dots,0) = f_0(f_1(a) \pmod 2) \neq f_0(f_2(a) \pmod 2) = f(0,a,0,\dots,0)$. Hence, the function f is not symmetric, and this is a contradiction.

From Lemma 4, it follows that symmetric functions of class 6 are only functions 0, 1, 2, S_{012}, $f_0(S_{001}(x_1) + \ldots + S_{001}(x_n) \pmod 2)$, $f_0(S_{010}(x_1) + \ldots + S_{010}(x_n) \pmod 2)$ and $f_0(S_{011}(x_1) + \ldots + S_{011}(x_n) \pmod 2)$. Thus, we obtain the numbers $t_{s,n}(6)$ and $\tau_{s,n}(5)$.

The numbers of symmetric bases and pivotal sets consisting of n-ary functions, can easily be determined by using Theorem 21 and Theorem 24.

REFERENCES

[1] Bagyinski, J., Demetrovics, J.: *The Structure of the Maximal Linear Classes in Prime-valued Logics*, C.R. *Math. Rep. Acad. Sci. Canada, Vol. II (1980), No. 4, 209 - 213.*

[2] Яблонский, С.В.: Функциональные построения в к-значных логиках, Труды МИАН СССР, Т. 51, Москва (1958), 5 - 142.

[3] Крнич, Л.: Типы базисов алгебры логики, Glasnik mat. fiz. i astronom., V. 20, 1965, No. 1-2, 23 - 32.

[4] Lau, D.: *Die Maximalen Klassen von* $Pol_k\{(x,x+1 \bmod k) \mid x \in E_k\}$, *Rostock. Math. Kolloq.25 (1984), 23 - 30.*

[5] Lau, D.: *Submaximale Klassen von* P_3, *EIK, 18 (1982), 4/5, 227 - 243.*

[6] Miyakawa, M.: *Functional Completeness and Structure of Three-valued Logic I-Classification of* P_3, *Researches of Electrotechnical Laboratory, No. 717, 1 - 85 (1971), Ibaraki.*

[7] Miyakawa, M.: *Enumeration of Bases of Maximal Clones of Three-valued Logical Functions (I)-SD (Slupecki functions), L (linear functions) and* P_+ *(self-dual functions), Bull. Electrotechn. Lab., Ibaraki, Vol. 47, No. 8 (1983), 39 - 49.*

[8] Stojmenović, I.: *Classification of P and the Enumeration of Bases of* P_3, *Rev. of Res., Fac. of Sci., Math. Ser., Novi Sad, 14, 1 (1984), 73 - 80.*

[9] Stojmenović, I.: *A Classification of the Set of Linear Functions in Prime-valued Logics, Acta Sci. Math., Szeged, (to appear).*

[10] Stojmenović, I.: *Classifications of the Three-valued Logical Symmetric Functions, Glasnik matematički, Zagreb, to appear.*

[11] Стойменович, И.: Перечисление симметрических функций предполных классов трехзначной логики, Rev. of Res., Fac. of Sci., Math. ser., Novi Sad, 13 (1983), 375 - 387.

[12] Tošić, R.: *S-bases of Propositional Algebra, Publ. del Inst. math., T. 14(28), (1972), 139 - 148, Beograd, 1973.*

REZIME

PREBROJAVANJE BAZA SEMIDEGENERISANIH, LINEARNIH I SAMODUALNIH FUNKCIJA PROSTOZNAČNE LOGIKE

U radu su odredjene klase funkcija i klase baza i pivotalnih skupova za skup S_p samodualnih funkcija p-značne logike (p - prost broj). Za taj skup i za skupove L_p linearnih funkcija p-značne logike i T poludegenerisanih funkcija troznačne logike su odredjeni brojevi n-arnih i simetričnih n-arnih funkcija svake klase funkcija, brojevi baza, pivotalnih skupova, simetričnih baza i simetričnih pivotalnih skupova svakog ranga, koje se sastoje od n-arnih funkcija ili funkcija koje zavise od najviše n promenljivih.

Received by the editors March 12, 1985.