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ON k-SEMINETS

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ABSTRACT

k-Seminets $(T,L_1,\dots,L_k)$ are introduced in [1], as a generalization of k-nets [4-5]. They are closely related to a special orthogonal system of partial quasigroups [1], as well as to the special codes [8 - 12], and to r-designs [13]. The article gives a characterization of k-seminets by means of $(T,L,\|)$-type objects, where $T\neq\emptyset$, $L\subset P(T)\setminus\{\emptyset\}$, $\| \subseteq L^2$, and $\|$ satisfies the axiom of Euclidean parallelism. Since affine planes and affine Sperner spaces are the objects of the same type, they are thus special k-seminets [4 - 5], [2]. Finally, it is shown that besides k-seminets $(T,L,\|)$ which are affine planes and affine Sperner spaces, there are other k-seminets in which every two different points are collinear.

1.

k-Seminets are defined in [1] in the following way:

Let T be a nonempty set, and L a collection of some nonempty subsets of T. Let $L_1,\dots,L_k$, $k\in N\setminus\{1,2\}$, be a partition of L. The elements of T are said to be points, the elements of L are lines, and the sets $L_1,\dots,L_k$ are classes of lines. Then the object $(T,L_1,\dots,L_k)$ is said to be a

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k-seminet iff the following conditions are satisfied:

M1 The intersection of every two lines from different classes L_i, L_j, $i,j \in \{1,\ldots,k\}$, is an element or the empty set[1]; and

M2 Every point from T belongs to one and only one line from each class L_i, $i \in \{1,\ldots,k\}$.

Because of M2, the lines from the same class have no common point [2].

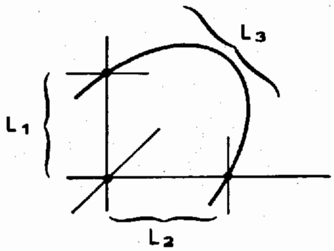

Fig. 1.

A 3-seminet with the least possible number of points is shown in Fig. 1.

If, instead of M1, we require:

M1′ Every two lines from different classes L_i, L_j, $i,j \in \{1,\ldots,k\}$. have exactly one common point;

then $(T, L, \ldots, L_k)$ is a k-net [4 - 5]. Every k-net is thus a k-seminet.

Comparing M1 and M1′ we get: If a k-seminet is not a k-net, then some lines from different classes have no common point. A parallelism of two lines in k-seminets is defined in [2] as a belonging to the same class L_i, $i \in \{1,\ldots,k\}$, and a skewness of two lines $p_1, p_2 \in L$ by $p_1 \cap p_2 = \emptyset$ and $p_1 \nparallel p_2$. In k-nets there are no skew lines. In that way defined, a binary relation of parallelism $\parallel$ is an RST-relation. It is easy to check that this relation satisfies the axiom of Euclidean parallelism, i.e. the proposition:

$$(\forall A \in T)(\forall p \in L)(\exists! p' \in L)(p' \parallel p' \wedge A \in p').$$

Considering M1, and the fact that, in general, not every two points in k-seminets (k-nets) are colinear, we find that the following proposition holds: for any two different points

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[1] *We prefer to say: two lines from different classes have at most one common point.*

there is at most one line to which they belong. And finally, M2 implies that in every k-seminet the following proposition is satisfied: for every point there is exactly $k \in N \setminus \{1,2\}$ different lines containing it. Thus we have proved the following proposition:

Proposition 1. *Let* $(T, L_1, \ldots, L_k)$ *be a* k-*seminet and let* $\|$ *be a binary relation on*

$$\mathcal{L} = \bigcup_{i=1}^{k} L_i$$

defined in the following way:

$$p \| q \overset{def}{\Leftrightarrow} p \in L_i \wedge q \in L_j \wedge i = j.$$

Then, the object $(T, \mathcal{L}, \|)$, *in which the elements of* T *are said to be points, the elements of* $\mathcal{L}$ *lines, and* $\|$ *is called the relation of parallelism, satisfies the following conditions:*

SR1 *Every point is contained in exactly* $k \in N \setminus \{1,2\}$ *different lines;*

SR2 *Any two different points are contained in at most one line;*

SR3 $\|$ *is an RST-relation; and*

SR4 $(\forall A \in T)(\forall p \cap \mathcal{L})(\exists! p' \in \mathcal{L})(p' \| p \wedge A \in p')$.

The following proposition also holds:

Proposition 2. *Let* T *be a nonempty set elements of which are said to be points, and let* $\mathcal{L}$ *be a nonempty collection of nonempty subsets of* T, *called lines. Let also* $\|$ *be a binary relation in* $\mathcal{L}$. *Now, if the object* $(T, \mathcal{L}, \|)$ *satisfies the conditions SR1 - SR4 from Proposition 1, then* $(T, L_1, \ldots, L_k)$, *with* $\{L_1, \ldots, L_k\} = \mathcal{L}/\|$, *is a* k-*seminet.*

Proof. a) By virtue of SR3, there is a partition of $\mathcal{L}$, namely

$$\{L_1,\ldots,L_t\} = L/\|.$$

b) Every class L_i, $i \in \{1,\ldots,t\}$, contains (in pairs) parallel lines. Thus, by *reductio ad absurdum*, using SR4, we conclude that different lines from the same class L_i, $i \in \{1,\ldots,t\}$, have no common points.

c) By SR1, every point A is contained in exactly $k \in N \setminus \{1,2\}$ lines, which, by virtue of b), belong to different classes $L_1,\ldots,L_k$. Hence, $t \geq k$. A conjecture that there is at least one more class - L_{k+1} contradicts SR4, and thus $t = k$. Thereby in the system $(T,L_1,\ldots,L_k)$, M2 holds.

d) In the system $(T,L_1,\ldots,L_k)$, M1 is also satisfied, because of SR2 and b).

The proposition is proved.

A direct consequence of Proposition 1 and Proposition 2 is the following proposition:

Theorem 3. *To every* k*-seminet* $(T,L_1,\ldots,L_k)$ *there corresponds an ordered triple* $(T,L,\|)$ *such that SR1 - SR4 are satisfied, and vice versa. In that correspondence* $\{L_1,\ldots$ $\ldots,L_k\} = L/\|$.

The meaning of Theorem 3 is the equalisation of k-seminet $(T,L_1,\ldots,L_k)$ and the object $(T,L,\|)$ in which SR1 - - SR4 hold. We shall thereby call such objects (satisfying SR1 - SR4) k-seminets.

In k-seminets $(T,L_1,\ldots,L_k)$ the skewness of two lines is defined by means of the previously defined parallelism in the following way [2]: p_1 and p_2 are skew iff $p_1 \cap p_2 = \emptyset$ and $p_1 \nparallel p_2$. In k-seminets $(T,L,\|)$ we define it directly.

A k-net $(T,L_1,\ldots,L_k)$ is a k-seminet in which M1′ holds instead of M1. Since this means that in k-nets there are no skew lines, we conclude that the following proposition holds:

Proposition 4. *To every* k-*net* $(T, L_1, \ldots, L_k)$ *there corresponds* k-*seminet* $(T, L, \parallel)$, *satisfying:*

SR5 *The set of pairs of skew lines is empty;*

and vice versa. In that correspondence $\{L_1, \ldots, L_k\} = L/\parallel$.

The meaning of Th. 4 is the equalisation of k-nets $(T, L_1, \ldots, L_k)$ and the objects $(T, L, \parallel)$ in which SR1 - SR5 are satisfied. Thereby, the object $(T, L, \parallel)$ satisfying SR1 - SR5 will be called k-net .

The following proposition is satisfied for every k-net $(T, L_1, \ldots, L_k)$ [4 - 5]:

SR6 $(\forall \ell \in L)(\forall \ell' \in L)|\ell'| = |\ell|(\geq 1)$.

Hence, using Proposition 4, we conclude that the following proposition holds:

Proposition 5. *Let* $(T, L, \parallel)$ k-*seminet. Then the following implication holds:*

$$SR5 \Rightarrow SR6.$$

The converse is not satisfied [2].

In the following consideration we assume that T is finite.

There are k-seminets satisfying the property:

SR2' Any two different points are contained in exactly one line[1]

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[1] *Any two different points are colinear.*

An affine Sperner space is introduced in [6][1]. It is defined in the following way:

Let T be an nonempty finite set, elements of which are said to be points, and L a collection of subsets of T, called lines. Let $\parallel$ be a binary relation on L. Then $(T,L,\parallel)$ is said to be an affine Sperner space (ASS) iff the propositions SR2′, SR3, SR4 and SR6 are satisfied. It is nontrivial iff $|L|\geq 2$[2].

It is shown in [2] that there is a bijection between nontrivial ASS and k-seminets $(T,L_1,\dots,L_k)$ in which SR2′ and SR6 are satisfied. This result can be formulated in the following way:

Corollary 6. *$(T,L,\parallel)$ is a nontrivial ASS iff $(T,L,\parallel)$ is a k-seminet, satisfying the propositions SR2′ and SR6.*

There is a bijection between the affine planes $(T,L,\parallel)$ and k-nets $(T,L_1,\dots,L_k)$ satisfying SR2′ [4 - 5]. Thereby, and by virtue of Proposition 4, we conclude that the following proposition holds:

Corollary 7. *$(T,L,\parallel)$ is an affine plane iff $(T,L,\parallel)$ is a k-seminet satisfying SR2′ and SR5.*

Remark 1. One can immediately see that the following proposition holds: $(T,L,\parallel)$ is an affine plane iff $(T,L,\parallel)$ is ASS in which the set of pairs of skew lines is empty. This proposition is here a direct consequence of Proposition 5, Corollary 6 and Corollary 7.

It might look like the colinearity of any two different points in k-seminets is a property only of those which are affine planes, i.e. of ASS. However, the following proposition holds:

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1) *[7], 293 - 294.*

2) *SR2′, SR3, SR4, SR6 are satisfied by the model in which $L=\{T\}$, $|T|\in N$, also [2].*

Proposition 8. *There are* k-*seminets* $(T, L, \parallel)$ *satisfying SR2′, and which are not* ASS[1].

The proposition proves, for example, a 6-seminet given in Fig. 2.

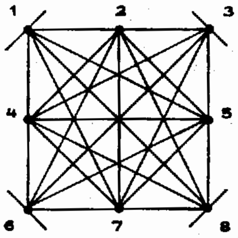

Fig. 2.

L_1 = {{1,2,3},{4,5},{6,7,8}};
L_2 = {{1,4,6},{2,7},{3,5,8}};
L_3 = {{1},{2,4},{3,6},{5,7},{8}};
L_4 = {{3},{2,5},{1,8},{4,7},{6}};
L_5 = {{1,5},{4,8},{2,6},{3,7}};
L_6 = {{1,7},{2,8},{3,4},{5,6}}.

Another example is a 3-seminet given in Fig. 1. This one can be constructed from an affine plane of order 2, by removing one of its points. In fact, one can get such k-seminets by removing one of its points.In fact,one can get such k-seminets by removing one point from any ASS.

Proposition 8 is a good reason to emphasise the following consequence of Corollary 6, and Corollary 7:

Proposition 9. *If any two different points in finite* k-*seminet* $(T, L, \parallel)$ *are colinear, and if this* k-*seminet is neither an affine plane nor an affine Sperner space, then its lines have not the same cardinality.*

Remark 2. In the definition of k-seminet $(T, L, \ldots, L_k)$ [1], the parallelism $\parallel$ is included in the axioms M1 and M2, since $\{L_1, \ldots, L_k\} = L/\parallel$ (Theorem 3). In the definition of k-seminet $(T, L, \parallel)$, however, there are axioms (SR1 and SR2) not including this notion.

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REZIME

O k-SEMIREŠETKAMA

k-Semirešetke $(T, L_1, \ldots, L_k)$ su uvedene u [1] kao jedna generalizacija k-rešetaka [4 - 5]. U tesnoj su vezi sa specijalnim ortogonalnim sistemima parcijalnih kvazigrupa [1], sa specijalnim kodovima [8 - 12] kao i sa r-dizajnima [13]. U ovom radu se nalazi jedna karakterizacija k-semireše-taka pomoću objekta tipa $(T,L,\parallel)$ gde je $T \neq \emptyset$, $L \subset P(T)\setminus\{\emptyset\}$, $\parallel \subseteq L^2$ a $\parallel$ zadovoljava „iskaz o euklidskoj paralelnosti". Pošto su afine ravni i afini prostori Spernera [6 - 7] objekti istog tipa, ovom su se direktno našli kao specijalne k-semi-rešetke [4 - 5], [2]. Na kraju se pokazuje da pored k-semire-šetaka $(T,L,\parallel)$ koje su afine ravni i afini prostori Spernera postoje i druge k-semirešetke u kojima je svaki par različi-tih tačaka kolinearan.

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