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COMMON FIXED POINTS FOR NONEXPANSIVE TYPE MAPPINGS IN CONVEX AND PROBABILISTIC CONVEX METRIC SPACES

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ABSTRACT

In this paper some common fixed point theorems in convex and probabilistic convex metric spaces are proved.

1. INTRODUCTION

Takahashi [1] introduced the notion of convexity in metric spaces and generalized some fixed point theorems in Banach spaces. Subsequently, Itoh [2], Tallman [3], Naimpally, Singh and Whitfield [4], Rhoades, Singh and Whitfield [5], and Hadžić [6] have studied convex metric spaces and fixed point theorems.

In this paper, we shall first introduce the concept of starshaped subsets in a convex and probabilistic convex metric space. Then we shall show some fixed point theorems for commuting mappings of the nonexpansive type on a starshaped subset of convex and probabilistic convex metric spaces. Our

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theorems improve and generalize the corresponding results in [6, 9, 10, 11].

2. DEFINITIONS AND LEMMAS

Definition 1. *Let* X *be a metric space and* $I = [0,1]$. *A mapping* $W : X \times X \times I \to X$ *is said to be a convex structure on* X *if for every* $(x,y,\lambda) \in X \times X \times I$ *and* $u \in X$

$$(1) \qquad d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y).$$

X *together with a convex structure* W *is called a convex metric space.*

In the following, we assume throughout that X is a convex metric space.

Definition 2. *A nonempty subset* K *of* X *is said to be convex if for each* $(x,y,\lambda) \in X \times X \times I$, $W(x,y,\lambda) \in K$.

Definition 3. *A nonempty subset* K *of* X *is called starshaped if there exists a* $x_0 \in K$ *such that the set* $\{W(x,x_0,\lambda) : x \in K, \lambda \in I\} \subset K$. *The* x_0 *is said to be a star-centre of* K.

Clearly, every convex set is a starshaped set and the inverse is not true.

Definition 4. *Let* K *be a starshaped subset of* X *with a star-centre* x_0. *We say that* K *satisfies condition* (β) *if for all* $(x,y,\lambda) \in K \times K \times I$

$$d(W(x,x_0,\lambda),W(y,x_0,\lambda)) \le \lambda d(x,y).$$

Definition 5. *A mapping* $S : X \to X$ *is said to be* (W,x_0)*-convex if for each* $(x,\lambda) \in X \times I$, $W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda))$.

Lemma 1. *Let* K *be a starshaped subset of* X *with a star-centre* x_0 *and let* $S : K \to K$ *be a* (W,x_0)*-convex mapping. Then the fixed point set* fix (S) *of* S *is a starshaped set with a star-centre* x_0.

Proof. By (1) we have $x_0 = W(x_0,x_0,\lambda)$, $\forall\lambda \in I$ $W(Sx_0,x_0,0) = x_0$. It follows from definition 5 that $Sx_0 = S(W(x_0,x_0,\lambda)) = W(Sx_0,x_0,\lambda)$ $\forall\lambda \in I$. Putting $\lambda = 0$, we obtain $x_0 = Sx_0$ and so $x_0 \in fix(S)$. Now for any $x \in fix(S)$ and $\lambda \in I$, we have

$$W(x,x_0,\lambda) = W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda)).$$

Thus the set $\{W(x,x_0,\lambda) : x \in fix(S), \lambda \in I\} \subset fix(S)$, i.e. $fix(S)$ is a starshaped set with the star-centre x_0.

Definition 6. *A continuous mappings* $f : K \to K$ *is said to be* α*-densifying if for any bounded* $D \subset K$ *with* $\alpha(D)>0$,

$$\alpha(f(D)) < \alpha(D),$$

where α *is the Kuratowski measure of noncompactness.*

For terminologies, notations and properties of Menger spaces (X,F,t), the reader may consult [7].

Definition 7. *Let* (X,F,t) *be a Menger space. A mapping* $W : X \times X \times I \to X$ *is said to be a convex structure if for all* $(x,y,u,\lambda) \in X \times X \times X \times (0,1)$,

$$F_{u,W(x,y,\lambda)}(2\varepsilon) \geq t(F_{u,x}(\tfrac{\varepsilon}{\lambda}),F_{u,y}(\tfrac{\varepsilon}{1-\lambda})),$$

for each $\varepsilon \geq 0$ *and* $W(x,y,0) = y$, $W(x,y,1) = x$.

Definition 8. *A nonempty set* $K \subset X$ *is said to be a starshaped subset of a Menger space* (X,F,t) *with a convex struc-*

ture W *if there exists a* $x_0 \in K$ *such that the set* $\{W(x,x_0,\lambda) : x \in K, \lambda \in I\} \subset K$. *The* x_0 *is called a star--centre of* K.

Definition 9. *A starshaped subset K of Menger space* (X,F,t) *with a convex structure* W *satisfies condition* (β) *if for all* $(x,y,\lambda) \in X \times X \times (0,1)$,

$$F_{W(x,x_0,\lambda),W(y,x_0,\lambda)}(\lambda\varepsilon) \geq F_{x,y}(\varepsilon).$$

for every $\varepsilon \geq 0$.

Definition 10. *Let* (X,F,t) *be a Menger space with a convex structure* W. *A mapping* $S : X \to X$ *is said to be* (W,x_0)*--convex if for each* $(x,\lambda) \in X \times I$, $W(Sx,x_0,\lambda) = S(W(x,x_0,\lambda))$.

Using a similar argument as in Lemma 1, we can easily prove

Lemma 2. *Let* (X,F,t) *be a Menger space with a convex structure* W, *and let* K *be a starshaped subset of* X *with a star-centre* x_0. *If* $S : K \to K$ *is a* (W,x_0)*-convex mapping, then the fixed point set* $\mathcal{F}ix(S)$ *of* S *is a starshaped subset with a star-centre* x_0.

3. MAIN RESULTS

Theorem 1. *Let* K *be a closed starshaped subset of a complete convex metric space* (X,d) *with a star-centre* x_0 *and* K *satisfy condition* (β). *Suppose that* $f : K \to K$ *is a nonexpansive mapping, i.e. for each* $x,y \in K$

$$d(fx,fy) \leq d(x,y),$$

$f(K)$ *is bounded and there exists* $m \in N$ *(the set of all positive integers) such that* f^m *is* α-densifying on $\{W(x,x_0,\lambda) : x \in f(K), \lambda \in I\}$. *Then* f *has a fixed point in* K.

Proof. Let $\{k_n\}_{n\in N}$ be a sequence of real numbers from (0,1) such that $\lim_{n\to\infty} k_n = 1$ and for each $n \in N$, define

$$f_n x = W(fx, x_0, k_n), \quad \forall x \in K.$$

Since K satisfies condition (β), we have that for all $x,y \in K$

$$d(f_n x, f_n y) = d(W(fx,x_0,k_n), W(fy,x_0,k_n))$$

$$\leq k_n d(fx,fy) \leq k_n d(x,y).$$

By Banach's contraction theorem, for each $n \in N$, there exists $x_n \in K$ such that $x_n = f_n x_n$. Furthermore,

$$d(x_n, fx_n) = d(f_n x_n, fx_n) = d(W(fx_n,x_0,k_n), fx_n)$$

$$\leq k_n d(fx_n, fx_n) + (1-k_n) d(fx_n, x_0)$$

and since f(K) is bounded, it follows that

$$\lim_{n\to\infty} d(x_n, fx_n) = 0.$$

Since f is nonexpansive, we have

$$d(x_n, f^m x_n) \leq \sum_{k=0}^{m-1} d(f^k x_n, f^{k+1} x_n) \leq m d(x_n, fx_n)$$

and so

$$(2) \qquad \lim_{n\to\infty} d(x_n, f^m x_n) = 0.$$

Let us prove that the set $\{W(fx,x_0,\lambda) : x \in K, \ \lambda \in (0,1)\}$ is bounded.
This follows from the inequality

$$d(u, W(fx,x_0,\lambda)) \leq \lambda d(u,fx) + (1-\lambda) d(u,x_0)$$

for all $(u,x) \in K \times K$ and since f(K) is bounded. From $x_n = f_n x_n$,

$\forall n \in N$, we have that $\{x_n\}_{n\in N} \subset \{W(fx,x_o,\lambda) : x \in K, \lambda \in (0,1)\}$ and so the set $\{x_n\}_{n\in N}$ is bounded. Furthermore, for any $\varepsilon > 0$ we have from (2) that

$$\alpha(\{x_n\}_{n\in N}) \leq \alpha(B(f^m(\{x_n\}_{n\in N}),\varepsilon)) \leq \alpha(f^m(\{x_n\}_{n\in N}))+\varepsilon$$

where

$$B(A,\varepsilon) = \{x \in K : d(x,A) < \varepsilon\} \quad (A \subset K) \text{ (see [8])}$$

and so

$$\alpha(\{x_n\}_{n\in N}) \leq \alpha(f^m(\{x_n\}_{n\in N})).$$

This implies that $\alpha(\{x_n\}_{n\in N}) = 0$ and there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$. Let $\lim_{k\to\infty} x_{n_k} = x^*$. Then from

$$d(x^*,fx^*) \leq d(x^*,x_{n_k}) + d(x_{n_k},fx_{n_k}) + d(fx_{n_k},fx^*)$$

and (2), since f is continuous, it follows that $x^* = fx^*$.

Theorem 2. *Let* K *be a closed starshaped subset of a complete convex metric space* (X,d) *with a star-centre* x_o *and* K *satisfy condition (β). Suppose that* $f,g,S,T : K \to K$ *such that* S *and* T *commute with* f *(or* g*),* f(K) *(or* g(K)) *is bounded and the following conditions are satisfied*

(i) *There exists* $m \in N$ *such that* f^m *is* α*-condensing on* $\{W(x,x_o,\lambda) : x \in f(K), \lambda \in I\}$ *and for all* $x,y \in K$, $d(fx,gy) \leq (Sx,Ty)$,

(ii) S *and* T *are* (W,x_o)*-convex and continuous.*

Then there exists $x^* \in K$ *such that* $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. Let $\mathcal{F}ix(S,T)$ denote the set of common fixed points of S and T. Since S and T are (W,x_o)-convex, it follows from Lemma 1 that $x_o \in \mathcal{F}ix(S,T)$ and $\mathcal{F}ix(S,T) \subset K$ is a starshaped subset of X with a star-centre x_o. By the continuity S and T, $\mathcal{F}ix(S,T)$ is also closed. From (i) we have that for all $x,y \in \mathcal{F}ix(S,T)$

$$d(fx,gy) \le d(x,y)$$

and hence $fx = gx$ for all $x \in \mathcal{F}ix(S,T)$ and for all $x,y \in \mathcal{F}ix(S,T)$

$$d(fx,fy) \le d(x,y).$$

For each $x \in \mathcal{F}ix(S,T)$, since S and T commute with f, we have $fx = fSx = Sfx$ and $fx = fTx = Tfx$ and so $fx = gx \in \mathcal{F}ix(S,T)$. Hence, it follows from Theorem 1 that there exists x^* in $\mathcal{F}ix(S,T)$ such that $x^* = fx^*$ and so $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Remark 1. It is easy to check that Theorem 1 of [6] is a very special case of Theorem 2.

Theorem 3. *Let* K *be a closed starshaped subset of a linear space* X *with a translation invariant metric satisfying* $d(\lambda x + (1-\lambda)y, 0) \le \lambda d(x,0) + (1-\lambda)d(y,0)$ *and* (X,d) *is complete. Suppose that* $f, g, S, T : K \to K$ *such that* S *and* T *commute with* f *(or* g*),* $f(K)$ *(or* $g(K)$*) is bounded and the following conditions are satisfied*

(i) *There exists* $m \in N$ *such that* f^m *is* α*-densifying on* $\{W(x,x_0,\lambda) : x \in f(K), \lambda \in I\}$ *and for all* $x,y \in K$, $d(fx,gy) \le d(Sx,Ty)$.

(ii) S *and* T *are continuous such that for each* $(x,\lambda) \in X \times I$

$$S(\lambda x + (1-\lambda)x_0) = \lambda Sx + (1-\lambda)x_0,$$
$$T(\lambda x + (1-\lambda)x_0) = \lambda Tx + (1-\lambda)x_0,$$

where x_0 *is a star-centre of* K.

Then there exists $x^* \in K$ *such that* $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. We define the mapping $W : X \times X \times I \to X$ by

$$W(x,y,\lambda) = \lambda x + (1-\lambda)y.$$

It is easy to check that W is a convex structure on X and so X is a complete convex metric space. Hence Theorem 3 follows from Theorem 2.

Remark 2. Theorem 3 improves and generalizes Theorem 2 of [9] and the corresponding results in [10, 11].

Theorem 4. *Let* K *be a closed starshaped subset of a complete Menger space* X *with a convex structure* W *which satisfies condition (β). Suppose that* $f : K \to K$ *is such that* $f(K)$ *is probabilistic bounded (which means that* $\sup_{\varepsilon\geq 0} D_{f(K)}(\varepsilon) = 1$ *where* $D_{f(K)}(\varepsilon) = \sup_{r<\varepsilon} \inf_{p,q\in f(K)} F_{p,q}(r)$, $\varepsilon \geq 0$), *for some* $m \in N$, f^m *is precompact on the set* $\{W(x,x_o,\lambda) : x \in f(K),\ \lambda \in (0,1)\}$ *where* x_o *is a star-centre of* K *and for all* $x,y \in K$ *and* $\varepsilon \geq 0$

$$(3) \qquad F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon).$$

Then f *has a fixed point in* K.

Proof. *Let* $\{k_n\}_{n\in N}$ be a sequence satisfying the condition in Theorem 1, and for each $n \in N$, define $f_n x = W(fx, x_o, k_n)$ for all $x \in K$. Since K satisfies condition (β) from (3), we have that

$$F_{f_n x, f_n y}(k_n\varepsilon) = F_{W(fx,x_o,k_n),W(fy,x_o,k_n)}(k_n\varepsilon)$$

$$\geq F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon),$$

for all $x,y \in K$ and $n \in N$. Since $f(K)$ is probabilistic bounded it is easy to check that the set $\{W(fx,x_o,k_n) : x \in K\}$ is also probabilistic bounded for each $n \in N$. From Theorem 11.2.3 of [12] it follows that for each $n \in N$, there exists $x_n \in K$ such that $x_n = f_n x_n$. Furthermore,

$$F_{x_n,fx_n}(2\varepsilon) = F_{f_nx_n,fx_n}(2\varepsilon) = F_{W(fx_n,x_0,k_n),fx_n}(2\varepsilon)$$

$$\geq t(F_{fx_n,fx_n}(\tfrac{\varepsilon}{k_n}),F_{fx_n,x_0}(\tfrac{\varepsilon}{1-k_n})) = t(1,F_{fx_n,x_0}(\tfrac{\varepsilon}{1-k_n}))$$

$$= F_{fx_n,x_0}(\tfrac{\varepsilon}{1-k_n}).$$

Since $f(K)$ is probabilistic bounded, for each $z \in K$ we have

$$\inf_{n\in N} F_{fx_n,fz}(\varepsilon) \geq \sup_{r<\varepsilon}\ \inf_{p,q\in f(K)} F_{p,q}(r) = D_{f(K)}(\varepsilon)$$

and so

$$(4) \qquad \sup_{\varepsilon\geq 0}\ \inf_{n\in N} F_{fx_n,fz}(\varepsilon) = \sup_{\varepsilon\geq 0} D_{f(K)}(\varepsilon) = 1$$

Since

$$(5) \qquad F_{fx_n,x_0}(\tfrac{\varepsilon}{1-k_n}) \geq t\left(F_{fx_n,fz}\left(\frac{\varepsilon}{2(1-k_n)}\right),F_{fz,x_0}\left(\frac{\varepsilon}{2(1-k_n)}\right)\right)$$

using the continuity of t, relations (4) and (5), and relation $\lim_{n\to\infty} k_n = 1$, we obtain that $\lim_{n\to\infty} F_{fx_n,x_0}(\frac{\varepsilon}{1-k_n}) = 1$ and so for each $\varepsilon > 0$,

$$(6) \qquad \lim_{n\to\infty} F_{x_n,fx_n}(\varepsilon) = 1.$$

Since for each $n \in N$, $x_n = f_nx_n = W(fx_n,x_0,k_n) \in \{W(x,x_0,\lambda) : x \in f(K),\ \lambda \in (0,1)\}$ and f^m is precompact on $\{W(x,x_0,\lambda) : x \in f(K), \lambda \in (0,1)\}$, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $\lim_{k\to\infty} f^m x_{n_k} = x^* \in K$. By (3), for each $\varepsilon \geq 0$, each $n \in N$ and each $k \in N$, we have that

$$F_{f^kx_n,f^{k+1}x_n}(\varepsilon) \geq F_{x_n,fx_n}(\varepsilon)$$

and so

$$F_{x_n,f^m x_n}(\varepsilon) \geq t\Big(F_{x_n,fx_n}(\tfrac{\varepsilon}{2}),t(F_{x_n,fx_n}(\tfrac{\varepsilon}{2^2}),\ldots$$
$$\ldots, F_{x_n,fx_n}(\tfrac{\varepsilon}{2^{m-1}})\Big)\ldots\Big).$$

Hence, from $t(1,1) = 1$, the continuity of t and (6), we obtain that

(7) $$\lim_{n\to\infty} F_{x_n,f^m x_n}(\varepsilon) = 1, \quad \forall\varepsilon > 0.$$

The continuity of t, relation (7) and the inequality

$$F_{x_{n_k},x^*}(\varepsilon) \geq t\Big(F_{x_{n_k},f^m x_{n_k}}(\tfrac{\varepsilon}{2}),F_{x^*,f^m x_{n_k}}(\tfrac{\varepsilon}{2})\Big)$$

imply that $\lim_{k\to\infty} x_{n_k} = x^*$. From the inequality

$$F_{x^*,fx^*}(\varepsilon) \geq t\Big(F_{x^*,x_{n_k}}(\tfrac{\varepsilon}{2}),t\Big(F_{x_{n_k},fx_{n_k}}(\tfrac{\varepsilon}{4}),$$
$$F_{fx_{n_k},fx^*}(\tfrac{\varepsilon}{4})\Big)\Big)$$
$$\geq t\Big(F_{x^*,x_{n_k}}(\tfrac{\varepsilon}{2}),t(F_{x_{n_k},fx_{n_k}}(\tfrac{\varepsilon}{4}),F_{x_{n_k},x^*}(\tfrac{\varepsilon}{4}))\Big),$$

it follows that $F_{x^*,fx^*}(\varepsilon)=1$ for each $\varepsilon > 0$ and so $x^* = fx^*$.

Theorem 5. *Let* K *be a closed starshaped subset of a complete Menger space* X *with a convex structure* W *which satisfies condition (B). Suppose that* $f,g,S,T : K \to K$ *are such that* S *and* T *commute with* f *(or* g*),* f(K) *(or* g(K)*) is probabilistic bounded and the following conditions are satisfied:*

(i) *There exists* $m \in N$ *such that* f^m *(or* g^m*) is precompact on the set* $\{W(x,x_0,\lambda) : x \in f(K), \lambda \in (0,1)\}$ *(or* $\{W(x,x_0,\lambda) : x \in g(K), \lambda \in (0,1)\}$*) and for all* $x,y \in K$

(8) $$F_{fx,gy}(\varepsilon) \geq F_{Sx,Ty}(\varepsilon),$$

(ii) S *and* T *are continuous and* (W,x_0)*-convex*

Then there exists $x^* \in K$ *such that* $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Proof. Let $\mathcal{F}ix(S,T)$ be the set of common fixed points of S and T. Since S and T are (W,x_0)-convex, from Lemma 2, it follows that $x_0 \in \mathcal{F}ix(S,T)$ and $\mathcal{F}ix(S,T)$ is a starshaped subset of X with a star-centre x_0. By the continuity of of S and T, $\mathcal{F}ix(S,T)$ is also closed. From (8) we have that for all $x,y \in \mathcal{F}ix(S,T)$

$$F_{fx,gy}(\varepsilon) \geq F_{x,y}(\varepsilon), \text{ for all } \varepsilon \geq 0,$$

and hence for all $x \in \mathcal{F}ix(S,T)$, $fx = gx$ and so all $x,y \in \mathcal{F}ix(S,T)$

$$F_{fx,fy}(\varepsilon) \geq F_{x,y}(\varepsilon), \forall \varepsilon \geq 0.$$

Since S and T commute with f, for each $x \in \mathcal{F}ix(S,T)$ we have that $fx = fSx = Sfx$ and $fx = fTx = Tfx$ and so $fx \in \mathcal{F}ix(S,T)$. Thus, from Theorem 4 it follows that there exists $x^* \in \mathcal{F}ix(S,T)$ such that $x^* = fx^*$ and so $x^* = fx^* = gx^* = Sx^* = Tx^*$.

Remark 3. Theorem 5 improves and generalizes Theorem 2 of [6].

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REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE NEEKSPANZIVNIH PRESLIKAVANJA U KONVEKSNIM I VEROVATNOSNIM KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazane su neke teoreme o zajedničkoj nepokretnoj tački u konveksnim i verovatnosnim konveksnim metričkim prostorima.

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