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THE FRENET FORMULAE OF THE RIEMANN-OTSUKI SPACE

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ABSTRACT

In this paper the Frenet formulae of the Riemann-Otsuki space with respect to covariant and contravariant part of the connection are obtained.

INTRODUCTION

The basis of the theory of Otsuki spaces has been laid down by T. Otsuki and A. Moór. The metric used determined that the observed space is of Weyl-Otsuki's or of Riemann-Otsuki's kind. In this paper we shall consider the Riemann-Otsuki space and we shall determine the Frenet formulae with respect to the co-resp. contravariant part of the connection. According to the following observation, we get that only in the contravariant part of the connection the Frenet formulae of the $R\text{-}O_n$ space are different from the known Frenet formulae of Riemannian geometry. The difference came from the fact that in Otsuki spaces $D\delta^i_j \neq 0$ holds.

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In all Otsuki spaces we have, with respect to the local coordinates x^i of an n-dimensional differentiable manifold, an *a-priori* given tensor P^i_j such that $det\|P^i_j\| \neq 0$ holds and the inverse tensor Q^i_j exists so that $P^i_j Q^j_r = \delta^i_r$. In the metric Otsuki spaces the metric tensor g_{ij} ($det\|g_{ij}\| \neq 0$) is given so that in the W - O_n Weyl-Otsuki space $\nabla_k g_{ij} = \gamma_k g_{ij}$, but in the R-$O_n$ (Riemann-Otsuki space)

$$(0.1) \qquad \nabla_k g_{ij} = 0$$

holds. In Otsuki spaces the covariant differential of the tensor T^i_j is defined by

$$(0.2) \qquad DT^i_j = P^i_a P^b_j \bar{D} T^a_b = P^i_a P^b_j (\partial_k T^a_b + {}'\Gamma_r{}^a{}_k T^r_b - {}''\Gamma_b{}^r{}_k T^a_r) dx^k .$$

The Leibnitz formula *does not hold* for this differential. The differential $\bar{D}$ is the *basic* covariant differential. The different coefficients of the connection are characteristic of the Otsuki spaces, and here are

$$(0.3) \qquad \delta^i_{j|k} = {}'\Gamma_j{}^i{}_k - {}''\Gamma_j{}^i{}_k \neq 0 .$$

The coefficient of the connection ${}''\Gamma_j{}^i{}_k$ was determined from the relation (0.1) and the coefficients of connection ${}'\Gamma_j{}^i{}_k$ are got from

$$(0.4) \qquad \partial_k P^i_j + {}''\Gamma_a{}^i{}_k P^a_j - P^i_a {}'\Gamma_j{}^a{}_k = 0 .$$

This relation is known as *Otsuki's relation*.

In Otsuki spaces it is possible to determine the covariant differentials D and $\bar{D}$ with respect only to the corresp. contravariant part of the connection. So

$$(0.5) \qquad {}'DT^i_j := {}'\nabla_k T^i_j dx^k = (\partial_k T^i_j + {}'\Gamma_r{}^i{}_k T^r_j - {}'\Gamma_j{}^r{}_k T^i_r) dx^k$$

holds. For this basic covariant differential the Leibnitz formula holds. The basic covariant differential $''\bar{D}$ can be defined in the same way. It is characteristic that the basic covariant differential $'\bar{D}$ is identical in the case of contravariant indices with the basic covariant differential $\bar{D}$, and similarly in the case of covariant indices the basic covariant differential $''\bar{D}$ is identical with the basic covariant differential $\bar{D}$.

In the following we shall use the relations

$$(0.6)\qquad '\bar{D}g_{ij} = dg_{ij} - ('\Gamma_i{}^r{}_k g_{rj} + '\Gamma_j{}^r{}_k g_{ir})dx^k,$$

$$(0.7)\qquad ''\bar{D}g_{ij} = dg_{ij} - (''\Gamma_i{}^r{}_k g_{rj} + ''\Gamma_j{}^r{}_k g_{ir})dx^k = 0,$$

$$(0.8)\qquad '\bar{D}g^{ra} = - g^{ia}g^{jr}('\bar{D}g_{ij}),$$

$$(0.9)\qquad ''\bar{D}g^{ra} = 0.$$

1. THE FRENET FORMULAE WITH RESPECT TO THE CONTRAVARIANT COMPONENTS OF THE VECTORS

Let the point P of the curve C : $x^i = x^i(s)$ be given where s is the arch length parameter. In that point $\underset{0}{\gamma}^i := \frac{dx^i}{ds}$ are the components of the unit tangent vector $\underset{0}{\gamma}$. Applying the basic covariant differential $'\bar{D}$ on the relation

$$(1.1)\qquad g_{ij}\underset{0}{\gamma}^i\underset{0}{\gamma}^j = 1$$

using the Leibnitz formula and the symmetry of the tensor g_{ij} we get

$$(1.2)\qquad ('\bar{D}g_{ij})\underset{0}{\gamma}^i\underset{0}{\gamma}^j + 2g_{ij}\underset{0}{\gamma}^i('\bar{D}\underset{0}{\gamma}^j) = 0.$$

From relations (0.6) and (0.7), it follows that $'\bar{D}g_{ij} = - 2g_{r(i}\bar{D}\delta^r_{j)}$. Substituting it in relation (1.2) and using that $'\bar{D}\underset{0}{v}^j = \bar{D}\underset{0}{v}^j$, we get

$$g_{ij}\underset{o}{v}^{i}(\bar{D}\underset{o}{v}^{j} - \underset{o}{v}^{r}\bar{D}\delta^{j}_{r}) = 0.$$

Let $\underset{1}{v}^{i}$ be the components of the unit vector $\underset{1}{v}$ so that

$$\text{(1.3)}\qquad \text{a)}\quad \underset{1}{v}^{j} := \frac{1}{\underset{1}{\kappa}(s)}(\bar{D}\underset{o}{v}^{j} - \underset{o}{v}^{r}\bar{D}\delta^{j}_{r}), \underset{1}{\kappa} > 0;\quad \text{b)}\ g_{ij}\underset{1}{v}^{i}\underset{1}{v}^{j} = 1$$

holds. From the above relations we can see, that $\underset{1}{v} \perp \underset{o}{v}$ and

$$\text{(1.4)}\qquad \underset{1}{\kappa}(s) = \left(g_{rq}(\bar{D}\underset{o}{v}^{r} - \underset{o}{v}^{a}\bar{D}\delta^{r}_{a})(\bar{D}\underset{o}{v}^{q} - \underset{o}{v}^{b}\bar{D}\delta^{q}_{b})\right)^{\frac{1}{2}}$$

and

$$\text{(1.5)}\qquad \bar{D}\underset{o}{v}^{j} = \underset{1}{\kappa}\underset{1}{v}^{j} + \underset{o}{v}^{q}\bar{D}\delta^{j}_{q}$$

hold. (1.5) is the *first Frenet formula* of the basic covariant differential applied on the *contravariant* components of the vectors.

Applying now the basic covariant differential $'\bar{D}$ on relation (1.3.b) with a calculation like the one above we get that $g_{ij}\underset{1}{v}^{i}(\bar{D}\underset{1}{v}^{j} - \underset{1}{v}^{r}\bar{D}\delta^{j}_{r}) = 0$. This means that the vector $\underset{1}{v}$ is orthogonal to the direction of $\bar{D}\underset{1}{v}^{j} - \underset{1}{v}^{r}\bar{D}\delta^{j}_{r}$, and if $\underset{2}{v}^{i}$ denotes the components of the unit vector orthogonal on the plain determined by the vectors $\underset{o}{v}$ and $\underset{1}{v}$, it follows that

$$\text{(1.6)}\qquad \bar{D}\underset{1}{v}^{j} - \underset{1}{v}^{r}\bar{D}\delta^{j}_{r} = \alpha\underset{o}{v}^{j} + \underset{2}{\kappa}\underset{2}{v}^{j}$$

holds. Now we shall use that $\underset{o}{v} \perp \underset{2}{v}$ and apply the covariant differential $'\bar{D}$ on the relation $g_{ij}\underset{o}{v}^{i}\underset{1}{v}^{j} = 0$. A calculation like the one above with the substitution $'\bar{D}\underset{o}{v}^{j}$ from (1.5) and $'\bar{D}\underset{1}{v}^{j}$ from (1.6) according to (1.1) and (1.3 b) gives

$$\text{(1.7)}\qquad \alpha = -\underset{1}{\kappa}(s).$$

Substituting it in (1.6), we get

$$(1.8) \qquad \bar{D}\underset{1}{v}^j = -\underset{1}{\kappa}\underset{0}{v}^j + \underset{2}{\kappa}\underset{2}{v}^j + \underset{1}{v}^r\bar{D}\delta^j_r .$$

From this relation there is $\underset{2}{v}^i = \frac{1}{\underset{2}{\kappa}(s)}(\bar{D}\underset{1}{v}^i + \underset{1}{\kappa}\underset{0}{v}^i - \underset{1}{v}^r\bar{D}\delta^i_r)$ and

$$\underset{2}{\kappa}(s) = \left(g_{ij}(\bar{D}\underset{1}{v}^j + \underset{1}{\kappa}\underset{0}{v}^j - \underset{1}{v}^r\bar{D}\delta^j_r)(\bar{D}\underset{1}{v}^i + \underset{1}{\kappa}\underset{0}{v}^i - \underset{1}{v}^b\bar{D}\delta^i_b)\right)^{\frac{1}{2}} .$$

Now we can formulate

Lemma 1. *If* v *is the unit vector, then* $(\bar{D}v^j - v^r\bar{D}\delta^j_r)g_{ij}v^i = 0$ *and* $(\bar{D}v_j - v_r\bar{D}\delta^r_j)g^{ij}v_i = 0$ *holds.*

(The proof of the covariant case follows in §2.)

We shall now make the following generalization.

Let for mutually orthogonal unit vectors $\underset{\ell}{v}$ ($\ell = 0,\dots,p-1$) be

$$(1.9) \qquad \bar{D}\underset{\ell}{v}^j = -\underset{\ell}{\kappa}\underset{\ell-1}{v}^j + \underset{\ell+1}{\kappa}\underset{\ell+1}{v}^j + \underset{\ell}{v}^r\bar{D}\delta^j_r$$

so that $\underset{0}{\kappa} = 0$, and if $q = 1,\dots,p-1$ then

$$(1.10) \qquad \underset{q}{\kappa} = \left(g_{ij}(\bar{D}\underset{q-1}{v}^j + \underset{q-1}{\kappa}\underset{q-2}{v}^j - \underset{q-1}{v}^r\bar{D}\delta^j_r)(\bar{D}\underset{q-1}{v}^i + \underset{q-1}{\kappa}\underset{q-2}{v}^i - \underset{q-1}{v}^t\bar{D}\delta^i_t)\right)^{\frac{1}{2}}$$

holds. We construct the vector $\underset{p}{v}^i$ ($p < n$) so that

$$(1.11) \qquad \underset{p}{v}^i := \frac{1}{\underset{p}{\kappa}}(\bar{D}\underset{p-1}{v}^i + \underset{p-1}{\kappa}\underset{p-2}{v}^i - \underset{p-1}{v}^r\bar{D}\delta^i_r) .$$

According to the above Lemma $g_{ij}\underset{p}{v}^i(\bar{D}\underset{p}{v}^i - \underset{p}{v}^r\bar{D}\delta^i_r) = 0$ holds and we can to write the linear combination

$$(1.12) \qquad \bar{D}\underset{p}{v}^j - \underset{p}{v}^r\bar{D}\delta^j_r = \alpha_0\underset{0}{v}^j + \alpha_1\underset{1}{v}^j + \dots + \alpha_{p-1}\underset{p-1}{v}^j + \underset{p+1}{\kappa}\underset{p+1}{v}^j .$$

Contracting this with $g_{ij}\underset{\ell}{v}^{i}$, using that $\underset{\ell}{v} \perp \underset{m}{v}$ $(m \neq \ell)$ and $\underset{\ell}{v}$ is the unit vector, we get

$$\alpha_{\ell} = g_{ij}\,\underset{\ell}{v}^{i}(\bar{D}\,\underset{p}{v}^{j} - \underset{p}{v}^{r}\bar{D}\delta_{r}^{j}).$$

With respect to (1.9) $\alpha_{\ell} = -g_{ij}\underset{\ell+1}{v}^{i}\,\underset{p}{v}^{j}\,\underset{\ell+1}{\kappa}$ i.e. if $\ell \neq p-1$, then $\alpha_{\ell} = 0$, and if $\ell = p-1$, then $\alpha = -\underset{p}{\kappa}$ holds. Substituting this in (1.12), it follows

$$\bar{D}\underset{p}{v}^{j} = -\underset{p}{\kappa}\,\underset{p-1}{v}^{j} + \underset{p+1}{\kappa}\,\underset{p+1}{v}^{j} + \underset{p}{v}^{q}\bar{D}\delta_{q}^{j} \tag{1.13}$$

and we can formulate

Theorem 1. *If* C : $x^{i}(s)$ *is the curve of an* R - 0_n *space and* $\underset{\ell}{v}, \ell = 0,\dots,p-1$, $(p < n)$ *are mutually orthogonal unit vectors which satisfy the relation (1.9) and* $\underset{p+1}{v}$ *is the unit vector orthogonal to all before and* $\underset{0}{\kappa} = 0$, $\underset{n}{\kappa} = 0$ *holds, then the vector* $\underset{p}{v}$ *satisfies the relation (1.9), too.*

If we use Otsuki's covariant differential D, then from the connection $Dv^{j} = P_{a}^{j}\bar{D}v^{a}$ it follows that $\bar{D}v^{a} = Q_{i}^{a}Dv^{i}$. Applying this on (1.9), we get

$$D\underset{\ell}{v}^{j} = P_{i}^{j}(-\underset{\ell}{\kappa}\,\underset{\ell-1}{v}^{i} + \underset{\ell+1}{\kappa}\,\underset{\ell+1}{v}^{i}) + \underset{\ell}{v}^{q}Q_{q}^{b}D\delta_{b}^{j} \tag{1.14}$$

with respect to $\ell = 0,\dots,(n-1)$; $\underset{0}{\kappa} = 0$; $\underset{n}{\kappa} = 0$. We can now state.

Theorem 2. *If in the point* M *of the curve* C *in the* R - 0_n *space the mutually orthogonal unit vectors* $\underset{0}{v},\underset{1}{v},\dots,\underset{n-1}{v}$ *satisfying relations (1.9) and (1.10) so that* $\underset{0}{\kappa} = 0$ *and* $\underset{n}{\kappa} = 0$ *hold, then (1.14) is the Frenet formula of the curve* C *of the* R - 0_n *space. This formula is applied with respect to the covariant differential* D *on the contravariant components of the observed vectors.*

Remark 1. *The relation (1.14) is the Frenet formula with respect to the covariant differential* $'D$, *too, applied on the contravariant components of the vectors.*

One can see that the difference between (1.14) and the known formula of Riemannian geometry is the covariant differential of the Kronecker-δ. In such a case but not only such special ones where this differential is zero, the formula (1.14) reduces to the known Frenet formula multiplied with the tensor P^i_j.

Now we shall apply the basic covariant differential $''\overline{D}$ on (1.1). Using the Leibnitz formula, the symmetry of the tensor g_{ij} and (0.7), we get that $g_{ij}\underset{0}{v}^i(''\overline{D}\underset{0}{v}^j) = 0$, i.e. $\underset{0}{v} \perp ''\overline{D}\underset{0}{v}$ holds. This means that we can construct the unit vector $\underset{1}{\overline{v}} \perp \underset{0}{v}$ so that

$$(1.15) \qquad ''\overline{D}\underset{0}{v}^j = \underset{1}{\kappa}^*\underset{1}{\overline{v}}^j,$$

where the scalar $\underset{1}{\kappa}^*(s)$ satisfies

$$(1.16) \qquad \underset{1}{\kappa}^*(s) = (g_{ij}\,''\overline{D}\underset{0}{v}^i\,''\overline{D}\underset{0}{v}^j)^{\frac{1}{2}} > 0.$$

The relation (1.15) is *the first Frenet formula* of the basic covariant differential $''\overline{D}$ applied on the contravariant components of the vectors. Now we shall prove

Theorem 3. *From the connection between the basic covariant differentials* $'\overline{D}$ *and* $''\overline{D}$ *it follows that* $\underset{1}{v} = \underset{1}{\overline{v}}$ *and the value of* $\underset{1}{\kappa}$ *is equivalent to the value of* $\underset{1}{\kappa}^*$.

Proof. It is easy to see that

$$(1.17) \qquad ''\overline{D}\underset{0}{v}^i = '\overline{D}\underset{0}{v}^i - \underset{0}{v}^q\overline{D}\delta^i_q$$

holds. Substituting this in (1.16) we get that according to (1.5) $\underset{1}{\kappa}(s) = \underset{1}{\kappa}^*(s)$ holds. In the following * by $\underset{1}{\kappa}(s)$ we shall denote that the curvature will be expressed with the aid of the basic covariant differential $''\bar{D}$.

According to the characteristics of the covariant differential $''D$, we can state

Theorem 4. *With respect to the basic covariant differential* $''\bar{D}$ *the Frenet formula of the curve* C *of the* $R - O_n$ *space is not different from the known formula of the Riemannian space. If* $\underset{o}{v}, \underset{1}{v}, \ldots, \underset{n-1}{v}$ *are in point* P *of curve* C *in a suitable way constructed mutually orthogonal unit vectors, then*

$$(1.18) \qquad ''D\underset{\ell}{v}^j = P^j_i\left(-\underset{\ell}{\kappa^*}\,\underset{\ell-1}{v}^i + \underset{\ell+1}{\kappa^*}\,\underset{\ell+1}{v}^i\right)$$

is the Frenet formula with respect to the covariant differential applied on the contravariant components of the observed vectors.

2. THE FRENET FORMULAE WITH RESPECT TO THE COVARIANT COMPONENTS OF THE VECTORS

According to the definition $\underset{o}{v}_i = g_{ij}\frac{dx^j}{ds}$ holds and $\underset{o}{v}_i$ are the covariant components of the unit tangent vector $\underset{o}{v}$. Applying the basic covariant differential $'\bar{D}$ on the relation $g^{ij}\underset{o}{v}_i\underset{o}{v}_j = 1$, using the Leibnitz formula and relations (0.8),(0.6) and (0.7), we get $g^{ij}\underset{o}{v}_i('\bar{D}\underset{o}{v}_j + \underset{o}{v}_b\bar{D}\delta^b_j) = 0$. This proved the second part of Lemma 1 from the first paragraph. Now again, as in the first paragraph, we can construct the mutually orthogonal unit vectors $\underset{\ell}{v}$ $(\ell = 0,1,\ldots,n-1)$, so that

$$(2.1) \qquad '\bar{D}\,\underset{\ell}{\bar{v}}_j = -\underset{\ell}{\kappa^{**}}\,\underset{\ell-1}{\bar{v}}_j + \underset{\ell+1}{\kappa^{**}}\,\underset{\ell+1}{\bar{v}}_j - \underset{\ell}{\bar{v}}_r\,\bar{D}\delta^r_j$$

holds with the remarks $\underset{o}{\kappa^{**}} = 0$, $\underset{n}{\kappa^{**}} = 0$, and if $\ell = 0,\ldots,n-2$ then

$$(2.2) \qquad \underset{\ell+1}{\kappa^{**}} = \left(g^{ij}\left('\bar{D}\underset{\ell}{\bar{v}}_j + \underset{\ell}{\kappa^{**}}\underset{\ell-1}{\bar{v}}_j + \underset{\ell}{\bar{v}}_r\bar{D}\delta^r_j\right)\left('\bar{D}\underset{\ell}{\bar{v}}_i + \underset{\ell}{\kappa^{**}}\underset{\ell-1}{\bar{v}}_i + \underset{\ell}{\bar{v}}_q\bar{D}\delta^q_i\right)\right)^{\frac{1}{2}}.$$

We can now formulate

Theorem 5. *From the relation* $\underset{\ell}{\bar{v}}_i = g_{ij}\underset{\ell}{\bar{v}}^j$ *it follows that the value of* $\underset{\ell}{\kappa}^{**}$ *is equal to the value of* $\underset{\ell}{\kappa}$ *and* $\underset{\ell}{\bar{v}} \equiv \underset{\ell}{v}$ *holds.*

Proof. If $\ell = 0$ then according to $\underset{o}{\bar{v}}_i \equiv \underset{o}{v}_i$ and $\underset{o}{v}_i = g_{ij}\underset{o}{v}^j$ using the Leibnitz formula from (2.2), we get

$$\underset{1}{\kappa}^{**} = \Big(g^{ij}(({}'\bar{D}g_{ja})\underset{o}{v}^a + g_{ja}({}'\bar{D}\underset{o}{v}^a) + g_{ra}\underset{o}{v}^a\bar{D}\delta^r_j)$$

$$(({}'\bar{D}g_{ib})\underset{o}{v}^b + g_{ib}({}'\bar{D}\underset{o}{v}^b) + g_{sb}\underset{o}{v}^b\bar{D}\delta^s_i)\Big)^{\frac{1}{2}}.$$

Since ${}'\bar{D}g_{ja} = -2g_{s(j}\bar{D}\delta^s_{a)}$ and (1.5) holds according to (1.3 b), it follows that $\underset{1}{\kappa}^{**} = \underset{1}{\kappa}$. Now we shall suppose that $\underset{\ell}{\kappa}^{**} = \underset{\ell}{\kappa}$ holds, and using the calculation as above, we get that

$$\underset{\ell+1}{\kappa}^{**} = (g_{ij}\underset{\ell+1}{\bar{v}}^i\underset{\ell+1}{\bar{v}}^j\underset{\ell+1}{\kappa}^2)^{\frac{1}{2}}$$

Since $\underset{\ell+1}{\bar{v}}^i$ are the unit vectors here, the statement of the first part of the theorem follows.

To prove the second part we shall use that $\underset{o}{v} \equiv \underset{o}{\bar{v}}$. Now we shall suppose that $\underset{p}{\bar{v}} \equiv \underset{p}{v}$ $(p = 0,\dots,\ell)$, and from (2.1), using the first part of the theorem, it follows that

$$\underset{\ell+1}{\bar{v}}_i = \frac{1}{\underset{\ell+1}{\kappa}}({}'\bar{D}\underset{\ell}{v}_j + \underset{\ell}{\kappa}\underset{\ell-1}{v}_j + \underset{\ell}{v}_r\bar{D}\delta^r_j).$$

Since $v_i = g_{ij}v^j$, using the Leibnitz formula and the calculation as above, contracting by g_{ia}, we get that

$$\underset{\ell+1}{\bar{v}}^r = \frac{1}{\underset{\ell+1}{\kappa}}({}'\bar{D}\underset{\ell}{v}^r + \underset{\ell}{\kappa}\underset{\ell-1}{v}^r - \underset{\ell}{v}^a\bar{D}\delta^r_a),$$

i.e. according to (1.11) $\underset{\ell+1}{\bar{v}}^r = \underset{\ell+1}{v}^r$ holds.

In the following $**$ by $\underset{\ell}{\kappa}$ we shall denote that the curvature will be expressed with the aid of the basic covariant differential ${}'\bar{D}$ applied on the covariant components of the vectors as in (2.2).

Theorem 6. *If in the point* M *of the curve* C *in the* R-O_n *space the mutually orthogonal unit vectors* $\underset{o}{v}, \underset{1}{v}, \dots$ $\dots \underset{n-1}{v}$ *are constructed so that* $\underset{p}{v}$ (p = 1,...,n-1) *satisfies*

$$(2.3) \qquad \underset{p}{v}_i := \frac{1}{\underset{p}{\kappa}^{**}}\left({}'\overline{D}\,\underset{p-1}{v}_i + \underset{p-1}{\kappa}^{**}\,\underset{p-2}{v}_i + \underset{p-1}{v}_r\,\overline{D}\delta^r_i\right)$$

with

$$\underset{o}{\kappa}^{**} = 0, \quad \underset{n}{\kappa}^{**} = 0,$$

then

$$(2.4) \qquad {}'D\underset{p}{v}_i = P^j_i\left(-\underset{p}{\kappa}^{**}\,\underset{p-1}{v}_i + \underset{p+1}{\kappa}^{**}\,\underset{p+1}{v}_i\right) - \underset{p}{v}_r D\delta^a_i Q^r_a$$

is the Frenet formula with respect to the covariant differential ′D *applied on the covariant components of the observed vectors.*

If we make the above calculation with respect to the basic covariant differential $''\overline{D}$, according to relation (0.9) and the fact that in the case of covariant indices the basic covariant differentials $\overline{D}$ and $''\overline{D}$ are not different, it follows that this case is not different from the observation of Riemannian space. We can only say

Remark 2. *The relation*

$$(2.5) \qquad D\underset{p}{v}_i = P^j_i\left(-\underset{p}{\kappa}^{***}\,\underset{p-1}{v}_i + \underset{p+1}{\kappa}^{***}\,\underset{p+1}{v}_i\right)$$

is the Frenet formula with respect to Otsuki's covariant differential D *applied on the covariant components of the vectors.*

Here ***, by scalars, denotes that the curvature will be expressed with the aid of Otsuki's basic covariant differential $\overline{D}$ applied on the contravariant components of the vectors.

From (2.5). it follows that

(2.6) $$\underset{p+1}{\kappa}^{***} = \left(g^{ij}(\bar{D}v_{\underset{p}{i}} + \underset{p}{\kappa}^{***} v_{\underset{p-1}{i}})(\bar{D}v_{\underset{p}{j}} + \underset{p}{\kappa}^{***} v_{\underset{p-1}{j}})\right)^{\frac{1}{2}}$$

holds. Using connection $\bar{D}v_i = {}'\bar{D}v_i + v_r\bar{D}\delta^r_i$, we get

$$\underset{p+1}{\kappa}^{***} = \left(g^{ij}({}'\bar{D}v_{\underset{p}{i}} + \underset{p}{\kappa}^{***} v_{\underset{p-1}{i}} + v_{\underset{p}{r}}\bar{D}\delta^r_i)({}'\bar{D}v_{\underset{p}{j}} + \underset{p}{\kappa}^{***} v_{\underset{p-1}{j}} + v_{\underset{p}{q}}\bar{D}\delta^q_j)\right)^{\frac{1}{2}}.$$

This means that the value of the curvatures is not different in the case of different differentials. This result can be expected because the curvature depends only on the curve and on a suitably constructed vector frame which is unequally determined. The Frenet formulae are different from the known formulae of Riemannian space only if $D\delta^i_j \neq 0$ holds, i.e. if the basic covariant differential of the metric tensor is not zero.

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REZIME

FRENETOVE FORMULE RIMAN-OTSUKIJEVOG PROSTORA

U radu su date Frenetove formule s obzirom na razne kovarijante diferencijale primenjene na kovarijante, odnosno na kontravarijantne indekse vektora.

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