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ON THE DISTRIBUTIVITY OF THE LATTICE OF L-VALUED SUBALGEBRAS OF FINITE ALGEBRAS

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ABSTRACT

The lattice $(S(A),\leq)$ of all L-valued (fuzzy) subalgebras of the given algebra A is considered. It is proved that for a finite algebra A, $(S(A),\leq)$ is isomorphic to a subdirect power of the lattice L, if $S(A)$ (the set of ordinary subalgebras of A) is closed under unions. Thereby, $(S(A),\leq)$ is distributive if and only if L is distributive. These results are applied to a class of groups.

1.

Let $A = (A,F)$ be a finite algebra, and $K \subseteq A$ a set of its constants. Let $(L,\wedge,\vee,0,1)$ be a bounded lattice (in the following denoted by L). An L-valued subset of (or: L-valued set on) A is a mapping $\overline{A} : A \to L$ (the set A and its subsets will be identified with their characteristic functions, where $0,1 \in L$; thus, $K(x) = 1$ if $x \in K$, othervise $K(x) = 0$). An L-valued subalgebra $\overline{B}$ of A is any L-valued subset of A, such that

a) $K \subseteq \overline{B}$, and

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b) $\overline{B}(f(x_1,\ldots,x_n)) \geq \overline{B}(x_1) \wedge \ldots \wedge \overline{B}(x_n)$, for all $x_1, \ldots, x_n \in A$, $f \in F_n \subseteq F$, $n \in N$.

$\overline{S(A)}$ is the set of all L-valued subalgebras of A, S(A) is the set of all ordinary subalgebras of A, and it is clear that $S(A) \subseteq \overline{S(A)}$.

$(\overline{S(A)},\leq)$ is a complete lattice (see, for example [4]).

Let $p \in L$, and $\overline{Ap} : A \to L$, where, for $x \in A$,

(1) $$\overline{Ap}(x) = \begin{cases} 1, & \text{if } x \in K \\ p, & \text{otherwise.} \end{cases}$$

Lemma 1. *If* $p \in L$, *then* $\overline{Ap} \in \overline{S(A)}$.

Proof. Obvious.

Proposition 2. *A mapping* $f : L \to S(A)$, *given by* $f(p) = Ap$ *is an embedding of the lattice* L *into* $(S(A),\leq)$.

Proof. By the definition of $\overline{Ap}$, f is one-to-one, and

$$f(p \wedge q) = \overline{Ap \wedge q} = \overline{Ap \wedge Aq},$$

since

$$(\overline{Ap} \wedge \overline{Aq})(x) = \overline{Ap}(x) \wedge \overline{Aq}(x) = \begin{cases} 1, & \text{if } x \in K \\ p \wedge q, & \text{if } x \notin K. \end{cases}$$

In exactly the same way one can prove that

$$\overline{Ap \vee p} = \overline{Ap} \vee \overline{Aq}.$$

Proposition 3. *Let* $A = (A,F)$ *be a finite algebra with the property that* $B, C \in S(A)$ *implies* $B \cup C \in S(A)$. *Then the lattice* $(\overline{S(A)},\leq)$ *is isomorphic to a sublattice of* $I^{|A|-|K|}$, *where this sublattice is a subdirect power.*

Proof. Let $A = \{a_1,\ldots,a_n,c_1,\ldots,c_k\}$, $K = \{c_1,\ldots$

$\ldots, c_k$, and let $f : \overline{S(A)} \quad L^{|A|-|K|}$, where $\overline{B} \to \overline{S(A)}$, and for $i = 1,\ldots,n$

$$f(\overline{B})(i) = \overline{B}(a_i).$$

Since for $c \in K$, $\overline{B}(c) = 1$, it follows that f is one-to-one. In the following, we identify $f(\overline{B})(i)$ with $\overline{B}(a_i)$, i.e. we consider $\overline{S(A)}$ as a subset of $L^{|A|-|K|}$. It is closed under the lattice operations:

Let $\overline{B},\overline{C} \in \overline{S(A)}$, and $x \in A \setminus K$. Then,

$$(B \sim C)(x) = B(x) \vee C(x) = \bigvee_{p\in L} p\cdot B_p(x) \vee \bigvee_{p\in L} p\cdot C_p(x) =$$

$$= \bigvee_{p\in L} p\cdot(B_p(x) \vee C_p(x)) = (\bigcup_{p\in L} p\cdot(B_p \cup C_p))(x),$$

by the definition of a L-valued union, and a decomposition property ([3]).

Since for every $p \in L$, $B_p \cup C_p \in S(A)$ (as it was required for A), it follows that $\bigcup_{p\in L} p\,(B_p \cup C_p) \in \overline{S(A)}$. Clearly, if $B_p \cup C_p = D_p \in \overline{S(A)}$, then

$$\bigcup_{p\in L} p\cdot D_p(x,y) \geq \bigcup_{p\in L} p\cdot(D_p(x) \wedge D_p(y)) =$$

$$= \bigvee_{p\in L} p\cdot D_p(x) \wedge \bigvee_{p\in L} p\cdot D_p(y).$$

Thus, $\overline{B} \cup \overline{C} = \overline{B} \vee \overline{C}$, and hence for every $x \in A$

$$(\overline{B} \vee \overline{C})(x) = \overline{B}(x) \vee \overline{C}(x).$$

Now, since $f(B \vee C) = f(B \cup C)$, it follows that for $i = 1,\ldots,n$

$$f(\overline{B} \vee \overline{C})(i) = f(\overline{B} \cup \overline{C})(i) = (\overline{B} \cup \overline{C})(a_i) =$$

$$= \overline{B}(a_i) \vee \overline{C}(a_i) = (f(\overline{B}) \vee f(\overline{C}))(i),$$

and thus (since $\overline{B} \cup \overline{C} = \overline{B} \vee \overline{C}$), $f(\overline{B} \vee \overline{C}) = f(\overline{B}) \vee f(\overline{C})$.

The proof that $\overline{S(A)}$ is closed under intersections is straightforward.

Thus we have proved that $(\overline{S(A)},\leq)$ is a sublattice of the lattice $L^{|A| - |K|}$. Moreover, it is a subdirect power, since for every $p \in L$, there is $\overline{B} \in \overline{S(A)}$ such that for $x \in A$, $\overline{B}(x) = p$. Namely, $\overline{B} = \overline{Ap}$ (defined by (1)).

Corollary 4. *Let* $A = (A,F)$ *be an algebra satisfying the conditions of Proposition 3. Then,* $(\overline{S(A)},\leq)$ *is a distributive lattice if and only if* L *is distributive.*

Proof. Let $(\overline{S(A)},\leq)$ be a distributive lattice. Then, by Proposition 2, L is distributive, as well.

If L is distributive, then $(\overline{S(A)},\leq)$ is also distributive, since it is a subdirect power of L, by Proposition 3.

Remark. The lattice of ordinary subalgebras $(S(A),\leq)$ of an algebra satisfying the conditions of Proposition 3 is distributive, since it is a sublattice of a distributive lattice $P(A) = B_2^A$ (B_2 is a Boolean algebra 2). Thus the properties of B_2 (used in meta-language) determine mainly the corresponding properties of the lattice $(S(A),\leq)$. Considering L-valued structures, i.e. taking the lattice L instead of B_2, one can see that properties are not always preserved.

THE CASE OF GROUPS

The preceding considerations, when applied to some classes of groups, can be formulated in a more concrete form.

Proposition 5. *Let* $(G,\cdot)$ *be a cyclic group of order* p^k, p-*prime*, $k \in N$, *and let* L *be any bounded lattice. Then,*

$$(\overline{S(G)},\leq) \cong (\{(q_1,\ldots,q_k) \mid q_1 \leq \ldots \leq q_k, q_j \in L\},\leq).$$

Proof. As it is shown in [1], all generators of the group G have the same value in L. The same is with the generators of every subgroup of G. Moreover, if g is a generator of G, and $\bar{H}$ an L-valued subgroup of G ($\bar{H} \in \overline{S(G)}$), then

$$\bar{H}(g) \leq \bar{H}(g^t),\ t \in N \quad \text{(see also [1])}.$$

There are k different subgroups, and thus k generators having different values in L. The proof now follows directly from Proposition 3.

REFERENCES

[1] M. Delorme: *Sous-groupes flous, Seminaire: Mathematique floue, Lion, 1978-79.*

[2] G. Grätzer: *General Lattice Theory, Akademie-Verlag, Berlin, 1978.*

[3] A. Kaufmann: *Introduction à la theorie des sous-ensembles flous, Paris, 1973.*

[4] G. Vojvodić, B. Šešelja: *On the lattice of L-valued subalgebras of an algebra, Zbornik radova PMF, Novi Sad (to appear).*

REZIME

O DISTRIBUTIVNOSTI MREŽE L-VREDNOSNIH PODALGEBRI KONAČNIH ALGEBRI

Razmatra se mreža svih L-vrednosnih (L je mreža sa 0 i 1) podalgebri date algebre. Dokazuje se da je za konačne algebre ta mreža izomorfna sa jednim poddirektnim stepenom od L (tačno odredjenog reda), pod uslovom da je skup običnih podalgebri te algebre zatvoren u odnosu na uniju. Dokazuje se da je mreža L-vrednosnih podalgebri distributivna ako i samo ako je L distributivna mreža. Za cikličke grupe reda p^k (p-prost, $k \in N$), daje se i konkretan opis te mreže.

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