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ON RANDOM BEST APPROXIMATIONS

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ABSTRACT

In this paper we prove a random generalization of S. Reich's theorem on best approximations. As a corollary we obtain a result on random fixed points.

1. INTRODUCTION

In [8] S. Reich proved the following theorem on best approximations.

Theorem A. *Let* S *be a nonempty compact and convex subset of a Banach space* $(E,\|\ \|)$ *and* $F : S \to 2^E$ *(the family of all nonempty subsets of* E*) a continuous multifunction such that* F(x) *is compact and convex for each* $x \in S$. *Then there exists an* $x \in S$ *such that:*

$$d(x,F(x)) = \inf_{(u,v) \in F(x)\times S} \|u - v\|.$$

V.M. Sehgal and S.P. Singh generalized Theorem A in the following way [10].

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Theorem B. *Let* $(E,\|\ \|)$ *be a Banach space,* (Ω,Σ) *a measurable space with* Σ *a* σ *algebra of subsets of* Ω *and* S *a nonempty compact and convex subset of* E. *If* $T : \Omega \times S \to 2^E$ *is a continuous random operator with compact and convex values, then there exists a measurable mapping* $g : \Omega \to S$ *satisfying:*

$$d(T(\omega,g(\omega)),g(\omega)) = \inf_{(u,v) \in T(\omega,g(\omega))\times S} \|u - v\|,$$

for each $\omega \in \Omega$.

In [10], as a corollary of Theorem A, a result on random fixed points is obtained.

We shall prove in this paper a similar result as Theorem B, for mappings T with a stochastic domain C.

2. PRELIMINARIES

In the paper we shall use the following notations: by (Ω,Σ) we shall denote a σ finite complete measure space, $(E,\|\ \|)$ a separable Banach space, $2^E = \{X \mid X \subseteq E,\ X \neq \emptyset\}$, $2^E_k = \{X \mid X \in 2^E$, X is compact$\}$ and $CB(E) = \{A \mid A \in 2^E$, A is closed and bounded$\}$.

A mapping $C : \Omega \to 2^E$ is *weakly measurable* if and only if for all open $D \subseteq E$:

$$C^{-1}(D) = \{\omega \mid \omega \in \Omega,\ C(\omega) \cap D \neq \emptyset\} \in \Sigma.$$

If a mapping $C : \Omega \to 2^E$ is such that $C(\omega)$ is compact, for every $\omega \in \Omega$, then C is weakly measurable if and only if $C^{-1}(G) \in \Sigma$, for every closed subset G of E [6]. A mapping $C : \Omega \to 2^E$ is *separable* [4] if and only if C is weakly measurable and there exists a countable set $Z \subseteq E$ such that for all $\omega \in \Omega$:

$$\overline{Z \cap C(\omega)} = C(\omega).$$

If $C : \Omega \to 2^E$ then $GrC = \{(\omega,x) \mid (\omega,x) \in \Omega \times E,\ x \in C(\omega)\}$.

Definition 1. [4] *Let* $C : \Omega \to 2^E$ *be a weakly measurable mapping. A mapping* $T : GrC \to 2^E$ *is called a random operator with a stochastic domain* C *if and only if for all* $x \in E$ *and open* $D \subset E$:

$$\{\omega | \omega \in \Omega,\ x \in C(\omega), T(\omega,x) \cap D) \neq \emptyset\} \in \Sigma$$

Definition 2. [4] *Let* $T : GrC \to 2^E$ *be a random operator with a stochastic domain* C. T *is called a continuous random operator if and only if for all* $\omega \in \Omega$, $T(\omega,\cdot)$ *is a continuous mapping from* $C(\omega)$ *into* $(CB(E),H)$, *where* H *is the Hausdorff metric on* CB(E).

Theorem C. [3] *Let* $f : GrC \to \mathbb{R}$ *be a continuous random function with a separable random domain* C *and* $a : \Omega \to \mathbb{R}$ *a measurable function. Then the mapping* $F : \omega \mapsto F(\omega)$ *is measurable, where:*

$$F(\omega) = \{x | x \in C(\omega),\ f(\omega,x) \leq a(\omega)\}, \qquad \omega \in \Omega.$$

Theorem D. [5] *Let* $C : \Omega \to 2^E$ *be a mapping with complete values. Then* C *is weakly measurable if and only if there exists a countable family* $U = \{u_1, u_2, \ldots\}$ *of measurable selectors for* C *such that:*

$$C(\omega) = \overline{U(\omega)} = \overline{\{u_n(\omega) | n \in \mathbb{N}\}}.$$

Theorem E. [7] *If* $F : \Omega \to E$ *is weakly measurable with complete values, then* F *has a measurable selector.*

3. A THEOREM ON BEST APPROXIMATIONS

The proof of the next lemma is similar to the proof of Lemma 2 from [10].

Lemma. *Let* $C : \Omega \to 2^E_k$ *and* $F : GrC \to 2^E_k$ *be a conti-*

nuous random operator. Then for every $\omega \in \Omega$ *the real functions:*

$$x \mapsto d(x,F(\omega,x)) \text{ and } x \mapsto d(C(\omega),T(\omega,x))$$

are continuous on $C(\omega)$.

Proof. From the inequality:

$$|d(x,F(\omega,x)) - d(y,F(\omega,y))| \leq D(F(\omega,x),F(\omega,y)) + \|x-y\|$$

for every $x,y \in C(\omega)$ and $\omega \in \Omega$ it follows that the mapping $x \mapsto d(x,F(\omega,x))$ is continuous on $C(\omega)$. Let for $\omega \in \Omega$:

$$g_\omega(x) = d(C(\omega),F(\omega,x)), \quad x \in C(\omega).$$

We shall prove the continuity of g_ω for every $\omega \in \Omega$.

Suppose that g_ω is not continuous at the point $x_0 \in C(\omega)$. Then there exists an $\varepsilon > 0$ and a sequence $\{x_n\}$ from $C(\omega)$ so that:

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad |g(x_n) - g(x_0)| > \varepsilon \ , \ n \in \mathbb{N}.$$

We shall prove that this leads to a contradiction. Since $F(\omega,x_0)$ and $C(\omega)$ are compact, there exist $a \in F(\omega,x_0)$ and $b \in C(\omega)$ so that:

$$d(C(\omega),F(\omega,x_0)) = \|a - b\| = \inf_{(u,v) \in C(\omega) \times F(\omega,x_0)} \|u - v\|.$$

Since the mapping $x \mapsto F(\omega,x)$ is continuous on $C(\omega)$, it is a lower semicontinuous mapping which implies the existence of a sequence $\{y_{n_i}\}$, $y_{n_i} \in F(\omega,x_{n_i})$, such that $\lim_{i\to\infty} y_{n_i} = a$. Then we have that:

$$d(C(\omega),F(\omega,x_{n_i})) \leq \|b - y_{n_i}\| \leq \|b - a\| + \|a - y_{n_i}\| =$$

$$= d(C(\omega),F(\omega,x_0)) + \|a - y_{n_i}\|$$

and so:

$$d(C(\omega),F(\omega,x_{n_i})) - d(C(\omega),F(\omega,x_0)) \leq \|a - y_{n_i}\|.$$

This implies that there exists $i_0(\varepsilon) \in \mathbb{N}$ so that $g(x_{n_i}) - g(x_0) < \varepsilon$, for every $i \geq i_0(\varepsilon)$. On the other hand let, for every $n \in \mathbb{N}$, $a_n \in C(\omega)$ and $y_n \in F(\omega,x_n)$ so that:

$$\|a_n - y_n\| = d(C(\omega),F(\omega,x_n)).$$

Since $C(\omega)$ is compact there exists a subsequence $\{a_{n_{i_r}}\}$ of $\{a_{n_i}\}$ such that $\lim_{r\to\infty} a_{n_{i_r}} = c$. Further, from the compactness of $F(\omega,x_0)$ and the uppersemicontinuity of $x \mapsto F(\omega,x)$ at $x = x_0$ there exists a subsequence $\{y_{n_{i_{r(k)}}}\}$ such that $\lim_{k\to\infty} y_{n_{i_{r(k)}}} = y \in F(\omega,x_0)$. Then we have that:

$$d(C(\omega),F(\omega,x_0)) \leq d(c,y) \leq d(c,a_{n_{i_{r(k)}}})+$$

$$+ d(a_{n_{i_{r(k)}}},y_{n_{i_{r(k)}}}) + d(y_{n_{i_{r(k)}}},y) = d(c,a_{n_{i_{r(k)}}}) +$$

$$+ d(C(\omega),F(\omega,x_{n_{i_{r(k)}}})) + d(y_{n_{i_{r(k)}}},y).$$

This implies that

$$d(C(\omega),F(\omega,x_0)) - d(C(\omega),F(\omega,x_{n_{i_{r(k)}}})) < \varepsilon$$

for $k \geq k_0(\omega)$. Hence there exists $k(\varepsilon)$ so that

$$|g(x_{n_{i_{r(k)}}}) - g(x_0)| < \varepsilon, \quad \text{for every } k \geq k(\varepsilon)$$

which is a contradiction.

Theorem. *Let* $C : \Omega \to 2^E_k$ *be a separable weakly measurable mapping such that* $C(\omega)$ *is convex, for every* $\omega \in \Omega$, $T : \mathrm{Gr}C \to 2^E_k$ *a continuous random operator such that* $T(\omega,x)$ *is convex for every* $(\omega,x) \in \mathrm{Gr}C$. *Then there exists a measurable* *mapping* $\varphi : \Omega \to E$ *such that for every* $\omega \in \Omega$, $\varphi(\omega) \in C(\omega)$ *and:*

$$d(\varphi(\omega),T(\omega,\varphi(\omega))) = d(C(\omega),T(\omega,\varphi(\omega))), \quad \omega \in \Omega.$$

Proof. Let $f : GrC \to \mathbb{R}$ be a real function defined by $f(\omega,x) = d(x,T(\omega,x)) - d(C(\omega),T(\omega,x))$, for every $\omega \in \Omega$ and $x \in C(\omega)$. Further, let for every $\omega \in \Omega$:

$$F(\omega) = \{x | x \in C(\omega), f(\omega,x) \leq 0\}.$$

Then $x \in F(\omega)$ implies that $f(\omega,x) = 0$, since for every $x \in C(\omega)$, $d(x,T(\omega,x)) \geq d(C(\omega),T(\omega,x))$. Hence, it remains to be proved that the mapping F has a measurable selector φ. In order to apply Theorem E we have to prove that the mapping F is measurable (it is obvious that $F(\omega)$ is closed and hence compact, for every $\omega \in \Omega$). We shall prove the measurability of F by using Theorem C, where $a(\omega) = 0$, for every $\omega \in \Omega$. Let us prove that f is a continuous random function. In [4] it is proved that the mapping $N_1 : GrC \to \mathbb{R}$, defined by

$$N_1(\omega,x) = d(x,T(\omega,x)), \ (\omega,x) \in GrC$$

is a continuous random function. We shall prove that the mapping $N_2 : GrC \to \mathbb{R}$ defined by:

$$N_2(\omega,x) = d(C(\omega),T(\omega,x)), \quad (\omega,x) \in GrC$$

is a continuous random function as well. In fact, we shall prove that for every $r > 0$ and every $x \in E$:

$$\{\omega | x \in C(\omega), N_2(\omega,x) \leq r\} \in \Sigma.$$

For every $s > 0$ and every $x \in E$:

$$\{\omega | \omega \in \Omega, x \in C(\omega), N_2(\omega,x) < s\} =$$

$$= \bigcup_{n\in N} \{\omega|\omega \in \Omega, x \in C(\omega), d(u_n(\omega),T(\omega,x)) < s\}$$

where $C(\omega) = \overline{\bigcup_{n\in N} u_n(\omega)}$ is the Castaing representation of C. Further [1], for every $n \in N$:

$$\{\omega|\omega \in \Omega, x \in C(\omega), d(u_n(\omega),T(\omega,x)) < s\} \in \Sigma$$

and so

$$\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) < s\} \in \Sigma.$$

If Z is a countable subset of E such that $\overline{Z \cap C(\omega)} = C(\omega)$ it is easy to prove that:

$$\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) \leq r\} =$$

$$= \bigcap_{n\in N} \bigcup_{z\in Z\cap L(x,1/n)} \{\omega|\omega \in \Omega, z \in C(\omega), N_2(\omega,z) < r+\frac{1}{n}\}$$

using the continuity of the mapping $d(C(\omega),T(\omega,\cdot))$ on $C(\omega)$, for every $\omega \in \Omega$. Hence, $\{\omega|\omega \in \Omega, x \in C(\omega), N_2(\omega,x) \leq r\} \in \Sigma$.

If φ is a measurable selector of F we have that $d(\varphi(\omega),T(\omega,\varphi(\omega))) = d(C(\omega),T(\omega,\varphi(\omega)))$, for every $\omega \in \Omega$.

Corollary. Let C and T be as in Theorem and $\overline{T(\omega,C(\omega))} \subseteq C(\omega)$, for every $\omega \in \Omega$. Then there exists a measurable mapping $\varphi : \Omega \to E$ so that $\varphi(\omega) \in C(\omega)$, for every $\omega \in \Omega$ and $\varphi(\omega) \in T(\omega,\varphi(\omega))$ (i.e. φ is a random fixed for the mapping T).

Proof. Since in this case we have that

$$d(C(\omega),T(\omega,\varphi(\omega))) = 0, \quad \text{for every } \omega \in \Omega$$

we have that $d(\varphi(\omega),T(\omega,\varphi(\omega))) = 0$ for every $\omega \in \Omega$.

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REZIME

O STOHASTIČKOJ NAJBOLJOJ APROKSIMACIJI

U ovom radu dokazana je stohastička generalizacija teoreme S. Reicha o najboljoj aproksimaciji. Kao posledica dobijen je rezultat o stohastičkim nepokretnim tačkama.

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