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A NOTE ON MAPPINGS AND PARACOMPACTNESS

Ilija Kovačević

Fakultet tehničkih nauka, Naučnoobrazovni institut za primenjene osnovne discipline, 21000 Novi Sad, V. Vlahovića 3, Jugoslavija

ABSTRACT

In the note some properties of paracompactness, α-regular and α-Hausdorff subsets, closed, almost closed and continuous mappings are observed.

1. PRELIMINARIES

Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed, unless explicitly stated.

A space X is *paracompact* iff every open covering of X has an open locally finite refinement, [3].

A space X is *almost paracompact* iff for every open covering $\mathcal{U}$ of X there exists an open locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ such that $X = \cup\{\overline{V} : V \in \mathcal{V}\}$, [10].

A subset A of a space X is *regularly open (regularly closed)* iff it is the interior (closure) of some closed (open) set, or equivalently iff it is the interior (closure) of its own closure (interior), [2].

A subset A of a space X is α*-paracompact (α-nearly*

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paracompact) iff for every X-open (X-regularly open) cover $\mathcal{U}$ of A there exists an X-open X-locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A, [13], ([8]).

A subset A of a space X is *α-almost paracompact* iff for every X-open cover of A there exists an X-locally finite family of X-open sets which refines it and the X-closures of whose members cover the set A, [5]. A subset A of a space X is *α-nearly compact* iff every X-regularly open cover of A has a finite subcovering, [11].

A subset A of a space X is *α-Hausdorff* iff any two points a,b of X, where $a \in A$ and $b \in X \setminus A$, can be strongly separated, [9]. A subset A of a space X is *α-regular* iff for any point $a \in A$ and any open set U containing a there exists an open set V such that $a \in V \subset \overline{V} \subset U$ or equivalently, for any closed set F of a space X and any point $x \in A$ such that $x \in X \setminus F$, there exist disjoint neighbourhoods of x and F, respectively, [9].

The multifunction $F : X \to Y$ is *closed (almost closed)* iff for any closed (regularly closed) set $A \subset X$, $F(A) = \cup \{F(x) : x \in A\}$ is closed in Y, [1], ([6]).

Theorem 1.1. ([7]) *Let* X *be a space such that there exists a dense α-regular subset* D. *If every* X-*open cover of* D *has an* X-*locally finite refinement which covers* D, *then* D *is α-paracompact, i.e.* X *is paracompact.*

Theorem 1.2. ([9]) *Let* A *be any α-regular α-paracompact subset of a space* X. *For each open neighbourhood* U *of* A, *there exists an open neighbourhood* V *of* A *such that* $A \subset V \subset \overline{V} \subset U$.

Theorem 1.3. ([5]) *A space* X *is almost paracompact iff for every regularly open covering* $\mathcal{U}$ *of* X *there exists an open locally finite family* $\mathcal{V}$ *which refines* $\mathcal{U}$ *such that* $X = \cup \{\overline{V} : V \in \mathcal{V}\}$.

Theorem 1.4. ([6]) *A surjective multifunction*

$F : X \to Y$ is almost closed iff for any subset $S \subset Y$ and any regularly open set U of a space X containing $F^-(S)$ $(F^-(S) = \{x \in X : F(x) \cap S \neq \emptyset\})$, there exists an open set V in Y such that $S \subset V$ and $F^-(V) \subset U$.

Theorem 1.5. ([6]) *Let a multifunction $F : X \to Y$ be an almost closed surjection such that $F^-(y)$ is α-nearly compact for each point $y \in Y$. If $\mathcal{U} = \{U_i : i \in I\}$ is an open locally finite family then $F(\mathcal{U}) = \{F(U_i) : i \in I\}$ is a locally finite family.*

2. SUBSETS AND PARACOMPACTNESS

Lemma 2.1. *Let A be any dense subset of a space X such that every open covering of A is an open covering of X. If X is paracompact then A is α-paracompact.*

Proof. Obvious.

In this lemma we supposed that every open covering of A is an open covering of X. If every open covering of a dense subset A is not an open covering of the space X, then A is not always α-paracompact, as can be seen from the following example.

Example 2.1. Let

$$X = \{a_i, b_i, a : i = 1,2,\ldots\} .$$

Let each point a_i be isolated. Let each point b_i be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a) : n = 1,2,\ldots\}$$

where

$$V^n(a) = \{a, a_i : i \geq n\} .$$

Let

$$A = \{a_i, b_i : i = 1,2,\ldots\}.$$

The space X is a Hausdorff regular paracompact. A is not α-paracompact.

Lemma 2.2. *Let* A *be any dense α-regular subset of a space* X *such that every open covering of* A *is an open covering of* X. *If* X *is almost paracompact, then* X *is paracompact.*

Proof. Let

$$\mathcal{U} = \{U_i : i \in I\}$$

be any open covering of X. For each point $x \in A$, there exists an open set V_x such that

$$x \in V_x \subset \overline{V}_x \subset U_i$$

for some $i \in I$.

Now,

$$\mathcal{V} = \{V_x : x \in A\}$$

is an open covering of A, i.e. of X. Since X is almost paracompact, there exists an open locally finite family

$$\mathcal{W} = \{W_j : j \in J\}$$

which refines $\mathcal{V}$ such that

$$X = \overline{\cup\{W_j : j \in J\}} = \cup\{\overline{W}_j : j \in J\}.$$

Now, for each $j \in J$, there exists an $x(j) \in A$ such that

$$W_j \subset V_{x(j)} \subset \overline{V}_{x(j)} \subset U_{i(x)}$$

for some $i(x) \in I$.
Hence, we have

$$\overline{W}_j \subset \overline{V}_{x(j)} \subset U_{i(x)} \, .$$

Now, the collection

$$\{\overline{W}_j \, : \, j \in J\}$$

is a locally finite family which refines $\mathcal{U}$ and covers A, hence A is α-paracompact, i.e. X is paracompact.

Theorem 2.1. *Let* A *be a dense* α*-regular subset of a space* X *such that every open covering of* A *is an open covering of* X. *If* X *is almost paracompact, then* A *is* α*-paracompact.*

Proof. It follows easily from Lemma 2.1 and Lemma 2.2.

Corollary 2.1. *Every regular almost paracompact is paracompact.*

There exists a space with the properties as in Lemma 2.2 which is not regular (Example 2.3 in [7]).

In Lemma 2.2 we supposed that every open covering of A is an open covering of X. If there is an open covering of a dense subset A which is not an open covering of X, then X need not be paracompact, as can be seen from the following example.

Example 2.2. Let

$$X = \{a_{ij}, a_i, a \, : \, i,j = 1,2,\ldots\}.$$

Let each point $a_{i,j}$ be isolated.
Let the fundamental system of neighbourhoods of a_i be the set

$$\{U^n(a_i) : n = 1,2,\ldots\}$$

where

$$U^n(a_i) = \{a_i, a_{ij} : j \geq n\} .$$

Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a) : n = 1,2,\ldots\}$$

where

$$V^n(a) = \{a, a_{ij} : i \geq n, j \geq n\} .$$

Then X is a Hausdorff space which is not regular at a and hence X is not paracompact (every Hausdorff paracompact space is regular). But X is almost paracompact. The subset

$$A = \{a_{ij}, a_i : i,j = 1,2,\ldots\}$$

is a dense α-regular subset of the space X. Not every open covering of A is an open covering of X (the set A is open).

Theorem 2.2. *Every α-Hausdorff α-almost paracompact subset of a space X is closed.*

Proof. Let A be any α-Hausdorff α-almost paracompact subset and a be any point of $X \setminus A$. For each $x \in A$, there exist open sets U_x and V_x such that

$$x \in U_x,\ a \in V_x \text{ and } U_x \cap V_x \neq \emptyset.$$

Now,

$$\mathcal{U} = \{U_x : x \in A\}$$

is an open covering of A. Since A is α-almost paracompact, then there exists an X-open X-locally finite family

$$\mathcal{W} = \{W_j : j \in J\}$$

which refines $\mathcal{U}$ such that

$$A \subset \overline{\cup\{W_j : j \in J\}} = \cup\{\overline{W}_j : j \in J\}.$$

Since $\mathcal{W}$ is X-locally finite, there exists an open neighbourhood U of a and a finite subset J_0 of J such that

$$U \cap W_j \neq \emptyset \text{ for } j \in J_0 \text{ and } U \cap W_j = \emptyset \text{ for } j \in J \setminus J_0.$$

For each $j \in J_0$ there is an $x(j) \in A$ such that $W_j \subset U_{x(j)}$. Let

$$U_1 = U \cap (\cap\{V_{x(j)} : j \in J_0\}).$$

U_1 is an open neighbourhood of a such that

$$a \in U_1 \subset X \setminus A$$

hence $X \setminus A$ is open, i.e. A is closed.

Corollary 2.2. *Let A be any dense α-almost paracompact α-Hausdorff subset of a space X. Then A = X, that is in the space X, there is no proper α-Hausdorff α-almost paracompact dense subset.*

3. MAPPINGS AND SUBSETS

Theorem 3.1. *Let f be a closed and continuous mapping of a space X onto a space Y and let A be a non-empty subset of X. If for each $x \in A \subset X$ the set $f^{-1}(f(x))$ is α-regular and α-paracompact, then f(A) is α-regular.*

Proof. Let $y \in f(A)$ and V be an open set containing

y. By Theorem 1.2 there exists an open set U in X such that

$$f^{-1}(y) \subset U \subset \overline{U} \subset f^{-1}(V).$$

Since f is closed, there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Now, we have

$$y \in W \subset f(U) \subset f(\overline{U}) \subset V.$$

Hence,

$$y \in W \subset \overline{W} \subset f(\overline{U}) \subset V$$

i.e. $f(A)$ is α-regular.

Corollary 3.1. *If f is a closed and continuous mapping of a regular space X onto a space Y such that $f^{-1}(y)$ is α-paracompact for each point $y \in Y$, then Y is regular.*

Problem 3.1. Let f be a closed and continuous mapping of a space X onto a space Y such that $f^{-1}(f(x))$ is α-paracompact for each $x \in A$. If A is α-regular, is $f(A)$ α-regular?

Theorem 3.2. *Let f be any almost closed mapping of a space X onto a space Y. Let B be a closed subset of X such that for each $x \in X \setminus B$ the set $f^{-1}(f(x))$ is α-regular and α-paracompact. Then $f(B)$ is closed.*

Proof. Let

$$y \in Y \setminus f(B), \text{ i.e. } f^{-1}(y) \subset X \setminus B.$$

By Theorem 1.2 there exists an open neighbourhood V of $f^{-1}(y)$ such that

$$f^{-1}(y) \subset V \subset \overline{V} \subset X \setminus B.$$

Since f is almost closed, there exists an open set W of Y such that

$$y \in W$$

and

$$f^{-1}(y) \subset f^{-1}(W) \subset \alpha(V) = \overline{V}^{\circ} \subset X \setminus B.$$

Hence, we have

$$y \in W \subset Y \setminus f(B)$$

and thus $Y \setminus f(B)$ is open, i.e. $f(B)$ is closed.

4. MAPPINGS AND PARACOMPACTNESS

Theorem 4.1. *Let* B *be any* α-*paracompact subset of* Y . *If* f *is a closed and continuous mapping of a space* X *onto a space* Y *such that for each* $y \in B$, $f^{-1}(y)$ *is* α-*paracompact, then* $f^{-1}(B)$ *is* α-*paracompact.*

Proof. *Let*

$$\mathcal{U} = \{U_i : i \in I\}$$

be any X-open covering of $f^{-1}(B)$. For each point $y \in B$, there exists an X-open X-locally finite family $\mathcal{V}_y$ which refines $\mathcal{U}$ such that

$$f^{-1}(y) \subset \cup\{V_y : V_y \in \mathcal{V}_y\} = V_y^*.$$

Since f is closed there exists an open set S_y of Y such that $y \in S_y$ and

$$f^{-1}(y) \subset f^{-1}(S_y) \subset V_y^*.$$

Now

$$S = \{S_y : y \in B\}$$

is a Y-open covering of B. Since B is α-paracompact, there exists a Y-open Y-locally finite family

$$K = \{K_j : j \in J\}$$

which refines S and covers B.
Now,

$$\{f^{-1}(K_j) : j \in J\}$$

is an X-open X-locally finite family which covers $f^{-1}(B)$. For each $j \in J$ there exists a $y_j \in B$ such that

$$K_j \subset S_{y_j}.$$

Hence

$$f^{-1}(K_j) \subset f^{-1}(S_{y_j}) \subset V^*_{y_j}.$$

For such y_j let

$$V = \{f^{-1}(K_j) \cap V_{y_j} : j \in J,\ V_{y_j} \in V_{y_j}\}.$$

The family V is an X-open covering of $f^{-1}(B)$ which refines U. It follows easily that V is X-locally finite, hence $f^{-1}(B)$ is α-paracompact.

Corollary 4.1. If f is a closed and continuous mapping of a space X onto a paracompact space Y such that for each point $y \in Y$ $f^{-1}(y)$ is α-paracompact, then X is paracompact.

Theorem 4.2. *If* F *is an almost closed upper semi-*

continuous and an open multifunction of an almost paracompact space X *onto a space* Y *such that for each* $y \in Y$ *the set* $F^{-}(y)$ *is* α*-nearly compact, for each point* $x \in X$ *the set* $F(x)$ *is* α*-nearly paracompact and for every proper closed subset* G *of* Y *the set* $F^{-}(G)$ *is a proper subset of* X*, then* Y *is almost paracompact.*

Proof. *Let*

$$\mathcal{U} = \{U_i : i \in I\}$$

be any regularly open covering of Y. Since for each $x \in X$ the set $F(x)$ is α-nearly paracompact and $\mathcal{U}$ is a regularly open covering of Y, there exists a Y-open Y-locally finite family

$$\{G_j : j \in J_x\}$$

which refines $\mathcal{U}$ such that

$$F(x) \subset \bigcup\{G_j : j \in J_x\}.$$

Let

$$G_x = \bigcup\{G_j : j \in J_x\}.$$

Thus G_x is an open set such that $F(x) \subset G_x$. Since F is upper semicontinuous, the collection

$$\{F^{+}(G_x) : x \in X\}$$

is an open covering of X. X is almost paracompact, so there exists an open locally finite family

$$\mathcal{V} = \{V_k : k \in K\}$$

which refines $\{F^{+}(G_x) : x \in X\}$ such that

$$X = \overline{\cup \{V_k : k \in K\}} = \cup \{\overline{V}_k : k \in K\}.$$

Next F is almost closed and open such that $F^-(y)$ is α-nearly compact, hence, by Theorem 1.5.

$$F(\mathcal{V}) = \{F(V_k) : k \in K\}$$

is a locally finite and open family.

For each $k \in K$ pick $x_k \in X$ such that

$$V_k \subset F^+(G_{x_k})$$

then $F(V_k) \subset F(F^+(G_{x_k})) \subset G_{x_k}$.

Let

$$\mathcal{V}^* = \{F(V_k) \cap G_j : j \in J_{x_k}, k \in K\}.$$

$\mathcal{V}^*$ is an open locally finite family which refines $\mathcal{U}$.

Now, we have to prove that

$$\bigcup_{k \in K} \bigcup_{j \in J_{x_k}} \overline{F(V_k) \cap G_j} = Y,$$

i.e.

$$\overline{\bigcup_{k \in K} \bigcup_{j \in J_{x_k}} F(V_k) \cap G_j} = \overline{\bigcup_{k \in K} F(V_k) \cap \bigcup_{j \in J_{x_k}} G_j} =$$

$$= \overline{\bigcup_{k \in K} F(V_k)} = \bigcup_{k \in K} \overline{F(V_k)} = Y.$$

Suppose that $\bigcup_{k \in K} \overline{F(V_k)}$ is a proper closed subset of Y.

There exists a point $x \in X$ such that

$F(x) \subset Y \setminus \bigcup_{k \in K} \overline{F(V_k)}$. Now,

$$F^+(Y \setminus \bigcup_{k \in K} \overline{F(V_k)}$$

is a nonempty open set such that

$$F^{+}(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap V_{k_j} \neq \emptyset$$

for some $k_j \in K$.

Let u be an element of X, such that

$$u \in F^{+}(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap V_{k_j}.$$

From $F(u) \subset Y \setminus \bigcup_{k\in K} \overline{F(V_k)}$ and $F(u) \subset F(V_{k_j})$, we have

$$(Y \setminus \bigcup_{k\in K} \overline{F(V_k)}) \cap F(V_{k_j}) \neq \emptyset .$$

This is a contradiction.

It follows that

$$\bigcup_{k\in K} \overline{F(V_k)} = Y.$$

Hence Y is almost paracompact.

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REZIME

O PRESLIKAVANJU I PARAKOMPAKTNOSTI

U radu se ispituju neke osobineparakompaktnosti, α-regularnih, α-Hausdorffovih podskupova, zatvorenih, skoro zatvorenih, neprekidnih preslikavanja, kao i višeznačno preslikavanje skoro parakompaktnog prostora.

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