ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 16,2(1986)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 16,2(1986)

A PAIR OF COMMUTING MAPPINGS WITH A COMMON FIXED POINT

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ABSTRACT

Two common fixed point theorems for two commuting mappings of a complete metric space into itself are given. These theorems generalize some earlier of Ćirić and the first author.

The first author [3], generalizing a result of Ćirić [1], defines a mapping T of a metric space (X,d) into itself to be a quasi-contraction if

(1) $$d(T^p x, T^q y) \le c\cdot max\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y): 0 \le r,r' \le p;\ 0 \le s,s' \le q\}$$

for all x,y in X, where $0 \le c < 1$ and p, q are some fixed positive integers.

AMS Mathematics Subject Classification (1980): Primary 47H10, Secondary 54H25.
Key words and phrases: Common fixed point, complete metric space, quasi-contraction.

The following result holds, see [3].

Theorem 1. *Let* T *be a continuous quasi-contraction of a complete metric space* (X,d) *into itself. Then* T *has a unique fixed point in* X.

An interesting generalization of this theorem was proved by Park and Rhoades [7] using a generalized version of a contractive condition of Hegedüs and Szilágyi [6].

Theorem 1 was also extended in [5] for a pair of continuous and commuting mappings S and T of a bounded, complete metric space into itself.

Here we give a further generalization of Theorem 1 for a pair of commuting mappings without requiring necessarily the simultaneous continuity of both S and T and by replacing the condition that X is bounded by the following condition

$$(2)\qquad \sup\{d(S^{r+i}T^{j}x, S^{r}T^{j'}x), d(T^{r+j}S^{i}x, T^{r}S^{i'}x) : r \geq 0,$$
$$0 \leq i, i' \leq p,\ 0 \leq j, j' < q\} = L < \infty,$$

for some particular x in X, p and q being fixed positive integers.

It is well known in the literature that the common fixed point of two mappings S and T is deduced as a limit of the sequence

$$(3)\qquad \{x, Tx, STx, \ldots, (ST)^{n}x, T(ST)^{n}x, \ldots\}.$$

Usually one first of all shows that the sequence (3) is bounded. As pointed out in [4], inequality (2) is certainly satisfied if X is bounded, but generally (2) does not imply that (3) is bounded if X is unbounded. To see this, consider $X = [0,\infty)$ with the euclidean metric and $Sx = x + 1$, $Tx = x + 2$ for all x in X. Then inequality (2), is satisfied, since $L = p + 2q - 3$, but the sequence (3) is unbounded.

Given a mapping T of (X,d) into itself, we now define the function $D_T(x)$ by

$$D_T(x) = d(x,Tx)$$

for all x in X. Motivated by a paper of Ćirić [2], we say that T has the property (α) in X if for any x_0 in X and for any sequence $\{x_n\}$ converging to x_0, we have

$$\lim_{n\to\infty} \sup D_T(x_n) \geq D_T(x_0).$$

Of course, any continuous mapping T has the property (α).

Theorem 2. *Let* S *and* T *be commuting mappings of a complete metric space* (X,d) *into itself, with* S *or* T *having property* (α), *satisfying the inequality*

$$(4) \qquad d(S^px,T^qy) \leq c\cdot max\{d(S^ix,S^{i'}x),d(T^jy,T^{j'}y),d(S^ix,T^jy): 0 \leq i,i' \leq p;\ 0 \leq j,j' \leq q\}$$

for all x, y *in* X, *where* $0 \leq c < 1$ *and* p, q *are fixed positive integers. Suppose further that (2) holds for some particular* x *in* X. *Then* S *and* T *have a unique common fixed point* z. *Further,* z *is the unique fixed point of* S *and* T.

Proof. Using inequality (4) we have for $r \geq p$ and $0 \leq j < q$

$$d(S^rT^jx,T^qx) \leq c\cdot max\ d(S^{r-i}T^jx,S^{r-i'}T^jx),d(T^{j'}x,T^{j''}x),$$

$$d(S^{r-i}T^jx,T^{j'}x) : 0 \leq i,i' \leq p;\ 0 \leq j',j'' \leq q\}$$

$$\leq c\cdot max\{L,L,d(S^{r-i}T^jx,S^rT^jx)+d(S^rT^jx,T^qx)+d(T^qx,T^{j'}x):$$

$$0 \leq i \leq p;\ 0 \leq j' \leq q\}$$

$$\leq c[2L + d(S^rT^jx,T^qx)]$$

on using (2). Thus

$$d(S^rT^jx,T^qx) \le 2Lc/(1 - c)$$

for $r \ge p$ and $0 \le j < q$. It follows that the set

$$\{S^rT^jx : r \ge 0;\ 0 \le j < q\}$$

is bounded. We can prove similarly that the set

$$\{T^rS^ix : r \ge 0;\ 0 \le i < p\}$$

is also bounded and it follows that

$$(5) \qquad sup\{d(S^rT^jx,S^nT^{j'}x),d(T^rS^ix,T^nS^{i'}x) : r,n \ge 0;\ 0 \le i,i' < p;\ 0 \le j,j' < q\} = K < \infty.$$

Let us now suppose that the set

$$A = \{S^{n-r}T^rx : n \ge 0;\ 0 \le r \le n\}$$

is unbounded. Then there exist integers r and n, with $n-r \ge p$, such that

$$(6) \qquad d(S^{n-r}T^rx,T^qx) > max\{Kc/(1-c),K\}$$

and

$$(7) \qquad d(S^{n-r}T^rx,T^qx) > max\{d(S^{m-i}T^ix,T^qx),d(S^{n-j}T^jx,T^qx) :\ 0 \le i \le m;\ 0 \le m < n;\ 0 \le j < r\}.$$

Now choose an integer k such that

$$(8) \qquad d(S^{n-r}T^rx,T^qx) > c^k\cdot max\{d(S^iT^jx,S^{i'}T^{j'}x) :\ 0 \le i,i',j,j' < n\}.$$

Using inequality (4) we have

$$d(S^{n-r}T^rx,T^q) \le c\cdot max\{d(S^{n-r-i}T^rx,S^{n-r-i'}T^rx),$$

$$d(T^jx,T^{j'}x),d(S^{n-r-i}T^rx,T^jx) :$$

$$0 \le i,i' \le p; \quad 0 \le j,j' \le q\}$$

$$\le c\cdot max\{d(S^{n-r-i}T^rx,S^{n-r-i'}T^rx),K,d(S^{n-r-i}T^rx,T^qx) +$$

$$+ d(T^qx,T^jx) : 0 \le i,i' \le p; \quad 0 \le j \le q\}$$

$$\le c\cdot max\{d(S^{n-r-i}T^rx,S^{n-r-i'}T^rx),$$

$$d(S^{n-r}T^rx,T^qx) + K : 0 \le i,i' \le p\}$$

because of (5) and (7).

Now

$$d(S^{n-r}T^rx,T^qx) \le c[d(S^{n-r}T^rx,T^qx) + K]$$

implies

$$d(S^{n-r}T^rx,T^qx) \le Kc/(1 - c),$$

contradicting inequality (6). We must therefore have

$$(9) \qquad d(S^{n-r}T^rx,T^qx) \le c\cdot max\{d(S^{n-r-i}T^rx,S^{n-r-i'}T^rx) :$$

$$0 \le i,i' \le p\},$$

where $r \ge q$, otherwise inequality (6) would again be contradicted. We can also omit all terms on the right-hand side of inequality (9) where $n-r-i < p$ and $n-r-i' < p$ because of (5) and inequality (6). Inequality (4) can therefore be applied to the remaining terms on the right-hand side of inequality (9) to give terms of the form

$$d(S^iT^jx, S^{i'}T^{j'}x)$$

where $0 \le i,i' \le n-r$ and $0 \le j,j' \le r$. Inequality (4) can be applied to these resulting terms either indefinitely or until terms of the form

$$(10) \qquad \{d(S^iT^jx, S^{i'}T^{j'}x) : 0 \le i,i' \le n-r;\ 0 \le j,j' < q\}$$

or

$$(11) \qquad \{d(S^iT^jx, S^{i'}T^{j'}x) : 0 \le i,i' < p;\ 0 \le j,j' \le r\}$$

or

$$(12) \qquad \{d(S^iT^jx, S^{i'}T^{j'}x) : 0 \le i \le n-r\ ;\ 0 \le i' < p\ ;\ 0 \le j \le r\ ;\ 0 \le j' < q\}$$

are obtained. Terms obtained after k applications of inequality (4) can be omitted because of inequality (8) and terms of the form (10) or (11) can be omitted because of (5). We must therefore have

$$\begin{aligned} d(S^{n-r}T^rx, T^qx) &\le c\cdot max\{d(S^iT^jx, S^{i'}T^{j'}x) : \\ &\qquad 0 \le i \le n-r;\ 0 \le i' < p;\ 0 \le j \le r;\ 0 \le j' < q\} \\ &\le c\cdot max\{d(S^iT^jx, T^qx) + d(T^qx, S^{i'}T^{j'}x) : 0 \le i \le n-r; \\ &\qquad 0 \le i' < p;\ 0 \le j \le r;\ 0 \le j' < q\} \\ &\le c[d(S^{n-r}T^rx, T^qx) + K] \end{aligned}$$

because of (5) and (7), again leading to a contradiction of inequality (6).

The set A therefore must be bounded and so

$$\sup\{d(S^nT^rx,S^{n'}T^{r'}x) : n,n',r,r' = 0,1,2,\ldots\} = M<\infty.$$

Without loss of generality, we will now suppose that $p \ge q$ and claim that

$$d((ST)^{np}u,(ST)^{np}v) \le c^n \cdot \max\{d(S^iT^ju,S^{i'}T^{j'}u),$$

(13) $$d(S^iT^ju,S^{i'}T^{j'}v),d(S^iT^jv,S^{i'}T^{j'}v) :$$

$$0 \le i,i',j,j' \le np\}$$

for $n = 0,1,2,\ldots$ and all u, v in X. This is certainly true when $n = 0$. Assume true for some n. Then

$$d((ST)^{(n+1)p}u,(ST)^{(n+1)p}v) =$$

$$= d((ST)^{np}(ST)^pu,(ST)^{np}(ST)^pv)$$

$$\le c^n \cdot \max\{d(S^iT^j(ST)^pu,S^{i'}T^{j'}(ST)^pu),$$

$$d(S^iT^j(ST)^pu,S^{i'}T^{j'}(ST)^pv),d(S^iT^j(ST)^pv,S^{i'}T^{j'}(ST)^pv)$$

$$0 \le i,i',j,j' \le np\}$$

$$\le c^{n+1} \cdot \max\{d(S^iT^ju,S^{i'}T^{j'}u),d(S^iT^ju,S^{i'}T^{j'}v),$$

$$d(S^iT^jv,S^{i'}T^{j'}v) : 0 \le i,i',j,j' \le (n+1)p\},$$

on using inequality (4). Inequality (13) follows by unduction.

Putting $u = (ST)^ix$ and $v = T(ST)^ix$ in inequality (13), we have

$$d((ST)^{np+i}x,T(ST)^{np+i}x) \le c^n \max\{d(S^iT^jx,S^{i'}T^{j'}x) :$$

$$0 \le i,i',j,j' \le (n+1)p\}$$

$$\le c^nM$$

for $i = 1,2,\dots,p$ and $n = 0,1,2,\dots$. We can prove similarly

$$d(T(ST)^{np+i-1}x,(ST)^{np+i}x) \le c^n M$$

for $i = 1,2,\dots,p$ and $n = 0,1,2,\dots$. It follows that the sequence (3) is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Now suppose that T has the property (α). Then we have

$$\limsup_{n\to\infty} D_T((ST)^n x) = \limsup_{n\to\infty} d((ST)^n x, T(ST)^n x)$$

$$= d(z,z) = 0$$

$$\ge D_T(z) = d(z,Tz)$$

and hence $Tz = z$.

Putting $u = S^i x$ and $v = x$ in inequality (13), it follows that

$$d((ST)^{np}S^i x,(ST)^{np}x) \le c^n M$$

for $i = 0,1,\dots,p$ and so the sequence

$$\{(ST)^{np}S^i x : n = 1,2,\dots\} \tag{14}$$

also converges to z for $i = 0,1,\dots,p$.

Further,

$$d((ST)^{np}S^p x,Sz) = d(S^p(ST)^{np}x,ST^q z) = d(S^p(ST)^{np}x,T^q Sz)$$

$$\le c\cdot \max\{d((ST)^{np}S^i x,(ST)^{np}S^{i'}x),d(T^j Sz,T^{j'}Sz),$$

$$d((ST)^{np}S^i x,T^j Sz) : 0 \le i,i' \le p;\ 0 \le j,j' \le q\}$$

$$= c\cdot \max\{d((ST)^{np}S^i x,(ST)^{np}S^{i'}x),d(Sz,Sz),$$

$$d((ST)^{np}S^i x,Sz) : 0 \le i,i' \le p\},$$

since T commutes with S. Letting n tend to infinity, it follows that

$$d(z,Sz) \leq cd(z,Sz)$$

and so z is also a fixed point of S. A similar proof can of course be given if one assumes that S has the property (α) instead of T. The uniqueness of z follows easily. This completes the proof of the theorem.

Remark 1. The example given [4] shows that the commutativity of S and T is necessary in Theorem 2.

Remark 2. Note that Theorem 2 is false if neither S nor T have the property (α). Indeed, let $X = [0,1]$ with the euclidean metric, let $S0 = 1/2$, $Sx = x/4$ if $x \neq 0$ and let $T0 = 1$, $Tx = x/2$ if $x \neq 0$. Then S and T commute and inequality (2) holds since X is bounded. Further, an easy calculation shows that inequality (4) is satisfied with $c = 1/4$, $p = 2$ and $q = 4$. Neither S nor T has the property (α) because for any sequence $\{x_n\}$, with $x_n \neq 0$ for any integer n, converging to 0

$$\limsup_{n\to\infty} D_S(x_n) = \limsup_{n\to\infty} \frac{3}{4}x_n = 0 < \frac{1}{2} = D_S(0)$$

and

$$\limsup_{n\to\infty} D_T(x_n) = \limsup_{n\to\infty} \frac{1}{2}x_n = 0 < 1 = D_T(0).$$

In the particular case that either $p = 1$ or $q = 1$, the condition that S or T has the property (α) is not necessary in Theorem 2. Indeed the following result holds.

Theorem 3. *Let* S *and* T *be commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$d(S^p x,Ty) \le c\cdot max\{d(S^i x,S^j x),d(y,Ty),d(S^i x,y), \tag{15}$$
$$d(S^i x,Ty) : 0 \le i,j \le p\}$$

for all x, y *in* X, *where* $0 \le c < 1$ *and* p *is a fixed positive integer. Suppose further that for some particular* x *in* X,

$$sup\{d(S^{r+i}x,S^r x),d(T^r S^i x,T^r S^j x) : \tag{16}$$
$$r \ge 0;\ 0 \le i,j < p\} < \infty \quad .$$

Then S *and* T *have a unique common fixed point* z. *Further,* z *is the unique fixed point of* S *and* T.

Proof. It follows as in the proof of Theorem 2 that the sequence (14) converges to a point z in X for $i = 0,1,\ldots,p$. Then using inequality (15) we have

$$d((ST)^{np}S^p x,Tz) \le c\cdot max\{d((ST)^{np}S^i x,(ST)^{np}S^i x),d(z,Tz),$$
$$d((ST)^{np}S^i x,z),d((ST)^{np}S^i x,Tz):$$
$$0 \le i,j \le p\}$$

and letting n tend to infinity it follows that

$$d(z,Tz) \le cd(z,Tz).$$

Thus $Tz = z$ and we then have $Sz = z$ as in the proof of Theorem 2.

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REZIME

PAR KOMUTIRAJUĆIH PRESLIKAVANJA SA ZAJEDNIČKOM NEPOKRETNOM TAČKOM

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Received by the editors May 15, 1986.