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ON A GENERALIZATION OF A CRAMER THEOREM*

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ABSTRACT

The paper extends the well-known theorem of Cramer concerning sufficient conditions for the stochastic process to have multiplicity equal to one.

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Let the stochastic process x(t) be given by the Cramer representation:

$$x(t) = \int_a^t g(t,u)dz(u), \tag{1}$$

$u \leq t$, $u,t \in (a,b) = T$, where z(u) is a process of orthogonal increments such that:

$$Ez(u) = 0, \quad \text{and} \quad Ez^2(u) = F(u),$$

and g(t,u), $u \leq t$, is a nonrandom function from space $L^2(dF(u))$:

$$\int_a^t g^2(t,u)dF(u) < \infty.$$

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The second order process $x(t)$, $t \in T$ is continuous to the left and purely nondeterministic.

Let us introduce the next conditions for $g(t,u)$ and $z(u)$:

R_1 The functions $g(t,u)$ and $g_t'(t,u)$ are bounded and continuous for $u \leq t$, $u,t \in T$.

R_2 $g(t,t) = 1$ for all $t \in T$.

R_3 The function $F(u) = Ez(u)$ is absolutely continuous and not identically constant and $f(u) = F'(u)$ has at most a finite number of discontinuity points in any finite subinterval of T.

Cramer in [2] proved the following statement:

Theorem. *The process* $x(t)$, $t \in T$ *from (1) which satisfies the conditions* R_1, R_2, R_3 *has a multiplicity one.*

Condition R_2 *and* $g(t,t) > 0$ *for all* $t \in T$, *are equivalent.*

It is easy to see that R_2 implies $g(t,t) > 0$ for all $t \in T$. Since $g(t,t) > 0$ for all $t \in T$, we may introduce the transformations:

$$\bar{g}(t,u) = g(t,u)/g(u,u),$$

$$\overline{dz}(u) = g(u,u)\cdot dz(u),$$

$u \leq t$, $u,t \in T$, such that R_2 holds for the process $x(t)$:

$$\bar{x}(t) = \int_a^t \bar{g}(t,u)\cdot \overline{dz}(u).$$

Remark. When $g(t,t) < 0$ for all $t \in T$, we may introduce the same transformations.

Cramer also proved that the upper conditions of the preceding theorem imply $x(t)$, $t \in T$ is given by a purely canonical representation (in Hida-Cramer mean), see theorem 5.2. in

[2]. But it is not valid. The following example shows it.

Example. Let x(t) be given by the expression:

$$x(t) = \int_0^t (-3t + 4u)dz(u),$$

$u \leqslant t$, $u,t \in (0,\tau) = T$, where $f(u) = F'(u) = 1$. Then, there exists a magnitude y:

$$y = \int_0^\tau u^2 dz(u),$$

from the Hilbert space H(x) generated by x(t), $t \in T$, such that the scalar product:

$$(x(t),y) = Ex(t)\cdot y = 0.$$

That means that the family $g(t,u) = -3t + 4u$ satisfying R_1 and R_2 is not complete in space $L^2(du)$, or x(t), $t \in T$ is not given by the Cramer representation.

If we omit condition R_2 from the preceding theorem, this gives us the main result.

Theorem. *The process* x(t), $t \in T$ *represented by (1) which satisfies the conditions* R_1 *and* R_2 *has multiplicity* $N = 1$.

Proof. Let us suppose that $N > 1$. For example let N be two. By condition R_1 the function $g(t,t) = (g_1(t,t), g_2(t,t))$, $t \in T$ may be equal to zero in a finite number of isolated points or on a finite number of subintervals of positive measures. Let us denote the union of intervals where $g_n(t,t) = 0$ by A_n and the union of intervals where $g_n(t,t) < 0$ by B_n, $n = 1,2$. From condition R_3 there exists a finite subinterval $T_1 \subset T$, $T_1 = (a_1, b_1)$ such that the derivatives $f_1(u)$ and $f_2(u)$ are continuous and not equal to zero. We choose such T_1 but we suppose, moreover, that T_1 does not contain the isolated zeros of the functions $g_1(t,t)$ and $g_2(t,t)$, and that T_1 is contained in

$\bigcap_{n=1}^{2} C_n$ where C_n is one of the next sets: A_n, B_n, $T \setminus (A_n \cup B_n)$, for n = 1,2. For instance, let us take $T_1 \subset A_1 \cap (T \setminus (A_2 \cup B_2))$. Then, on the interval T_1 : $g_1(t,t) = 0$ and $g_2(t,t) > 0$, for all $t \in T$.

Let t be any point in T_1 and let h(u) be a function in $L^2(dF_1(u) \times dF_2(u))$ such that the next relation is valid:

$$\sum_{n=1}^{2} \int_{a_1}^{s} h_n(u) g_n(s,u) f_n(u)du = 0,$$

$s < t$, $u < s$, $t \in T$. By condition R_1 the relation may be differentiated with respect to s, and we obtain:

$$\sum_{n=1}^{2} \int_{a_1}^{s} g'_{n_s}(s,u)\, h_n(u)\, f_n(u)\, du + \sum_{n=1}^{2} g_n(s,s)\, h_n(s)\, f_n(s) = 0,$$

for $s \in (a_1,t]$. This implies that:

$$\int_{a_1}^{s} g'_{1_s}(s,u)h_1(u)f_1(u)du + \int_{a_1}^{s} g'_{2_s}(s,u)h_2(u)f_2(u)du + g_2(s,s)h_2(s)f_2(s) = 0,$$

for $s \in (a_1,t]$. This equation is satisfied, if for example:

$$\int_{a}^{s} g'_{1_s}(s,u)h_1(u)f_1(u)du = 1,$$

and

$$\int_{a}^{s} g'_{2_s}(s,u)h_2(u)f_2(u)du + g_2(s,s)h_2(s)f_2(s) = -1.$$

These are nonhomogeneous integral equations of Volterra of the first and second type, and for each there exist bounded, continuous and not almost everywhere equal to zero solutions $f_1(s)h_1(s)$ and $f_2(s)h_2(s)$ for $s \in (a_1,t]$. By our hypothesis about $f_1(u)$ and $f_2(u)$ on T_1, it follows that:

$$(2) \qquad \int_a^t h_1^2(u)dF_1(u) + \int_a^t h_2^2(u)dF_2(u) > 0.$$

That means $g(t,u) = (g_1(t,u), g_2(t,u))$ is not complete in $L^2(dF_1(u) \times dF_2(u))$.

We can show in a similar way that (2) holds when T_1 is contained in one of the remaining eight sets.

When we suppose that N is equal to any natural number $N > 2$, the conclusion is the same as before.

The proof is completed.

The representation (1) of $x(t)$, $t \in T$ which satisfies the conditions R_1 and R_2 can be not purely canonical.

Example. Let the process $x(t)$ be given by:

$$x(t) = \int_0^t \left(\frac{3}{5} - 3\frac{u}{t} + 5\frac{u^2}{t^2} - 3\frac{u^3}{t^3} + \frac{2}{5}\frac{u^5}{t^5}\right)dz(u),$$

where $u \le t$, $u,t \in (0,\tau) = T$ and $f(u) = F'(u) = 1$.
The functions:

$$g(t,u) = \frac{3}{5} - 3\frac{u}{t} + 5\frac{u^2}{t^2} - 3\frac{u^3}{t^3} + \frac{2}{5}\frac{u^5}{t^5}$$

$$g_t'(t,u) = \frac{1}{t}\left(3\frac{u}{t} - 10\frac{u^2}{t^2} + 9\frac{u^3}{t^3} - 2\frac{u^5}{t^5}\right),$$

for $u \le t$ are bounded and continuous on TxT. From the fact: $g(t,t) = g_t'(t,t) = 0$ for all $t \in T$, follows the limitation of $g(t,u)$ and $g_t'(t,u)$ when $t \to 0$. The process $x(t)$, $t \in T$ has multiplicity equal to one. But this representation of $x(t)$, $t \in T$ is not purely canonical. There exists the magnitude $y \in H(x)$:

$$y = \int_0^\tau u^2 dz(u),$$

such that the following relation is valid:

$$\int_0^t \left(\frac{3}{5} - 3\,\frac{u}{t} + 5\,\frac{u^2}{t^2} - 3\,\frac{u^3}{t^3} + \frac{2}{5}\,\frac{u^5}{t^5}\right)\cdot u^2 du = 0.$$

Hence, the conditions R_1 and R_3 do not imply that the process $x(t)$, $t \in T$ is given by the Cramer representation (or purely canonical one).

The problem is now if we can omit the condition of a purely canonical representation of $x(t)$, $t \in T$ in the proofs of the preceding theorems?

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REZIME

JEDNO UOPŠTENJE KRAMEROVE TEOREME

Rad proširuje dobro poznatu teoremu Cramera koja se odnosi na dovoljne uslove da bi stohastički proces imao multiplicitet jednak jedan.