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# SOME RESULTS ON M- AND H- MATRICES

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## ABSTRACT

In this paper some sufficient conditions for a matrix to be an M- or an H- matrix are given. The results include the ones from [1].

## INTRODUCTION

We shall begin with some notations:

$$N = \{1,2,\ldots,n\},\ N(i) = N\setminus\{i\},\ i\in N.$$

For any matrix $A = [a_{ij}]\in C^{n,n}$ (= set of all the complex nxn matrices) and $i\in N$, $\alpha\in[0,1]$, we define

$$P_i(A) = \sum_{j\in N(i)} |a_{ij}|,\quad Q_i(A) = \sum_{j\in N(i)} |a_{ji}|,\quad R_i(A) = \sum_{j\in N} |a_{ij}|,$$

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$$P_{i,\alpha}(A) = \alpha\, P_i(A)+(1-\alpha)Q_i(A), \quad Q_i^*(A) = \max_{j\in N(i)} |a_{ji}|,$$

$$Q_i^{(r)}(A) = \max_{t_r\in e_r} \sum_{j\in t_r} |a_{ji}|,$$

where $r\in N$ and $e_r$ is the set of all the choices $t_r=\{i_1,\dots,i_r\}$ of different indices from N.

Let $e_i : R_+^{2p(i)} \to R_+$, $p(i)\in N$, $i=1,2$, be two functions satisfying the following condition

(1) $x \geq y \Rightarrow e_i(x) \geq e_i(y)$, for any $x,y\in R_+^{2p(i)}$, $i=1,2$.

For any $A\in C^{n,n}$, $s\in N$ and $J = \{j_1,\dots,j_k\} \subset N$ we define

$$e_i(A,J,s) = e_i(R_{j_1}(A),\dots,R_{j_k}(A),Q_{j_1}^{(s)},\dots,Q_{j_k}^{(s)}(A)), \quad i=1,2.$$

Let $s$ and J be fixed and $t_r\in e_r$. Then, we define

$$E(A,t_r) = e_1(A,t_r,s) + e_2(A,J,s).$$

Let $e_1^{(i)}$, $e_2^{(i)}$ and $E^{(k)}$ be functions of the above form and let $K(m,r) = \{(e_1^{(i)},e_2^{(i)}): i=1,\dots,m\}$. From now on we shall suppose that A=D-B, where D is diagonal part of A.

***Definition 1.*** A matrix A is called K(m,r)-diagonally dominant iff for each $t_r\in e_r$ there exists an index $k\in\{1,\dots,m\}$, such that

$$E^{(k)}(D,t_r) > E^{(k)}(B,t_r).$$

*Definition 2.* A set $K(m,r)$ is called K-regular iff any $K(m,r)$ -diagonally dominant matrix is regular.

*Definition 3.* A matrix A is called $K(r(1),\dots,r(m))$-diagonally dominant iff for each $j\in\{1,2,\dots,m\}$ and for each $t_{r(j)}\in e_{r(j)}$ it is fulfiled that

$$E^{(j)}(D,t_{r(j)}) > E^{(j)}(B,t_{r(j)}).$$

*Definition 4.* A set $K(m,r)$ is called $K_1$-regular iff any $K(r(1),\dots,r(m))$-diagonally dominant matrix is regular.

*Definition 5.* A real square matrix whose off-diagonal elements are all non-positive is called an L-matrix.

*Definition 6.* A regular L-matrix A, for which $A^{-1}\geq 0$ is called an M-matrix.

For $C = [c_{ij}]$ and $F = \mathrm{diag}(f_1,\dots,f_n)$, we shall write $C \geq 0$ iff $c_{ij} \geq 0$ for each $i,j\in N$ and $F > 0$ iff $f_i > 0$ for each $i\in N$.

## SUFFICIENT CONDITIONS FOR L-MATRICES TO BE M-MATRICES

*Theorem 1.* Let $K(m,r)$ be a K-regular set. Let A be an L-matrix, whose diagonal elements are all positive. If A is a $K(m,r)$-diagonally dominant, then it is an M-matrix.

P r o o f : $A = D-B$, $D > 0$, $B \geq 0$ and $D^{-1}A = I - D^{-1}B$ where I is identity matrix. Let us prove that $\rho(D^{-1}B) < 1$. We shall suppose that there exists an eigenvalue $\lambda$ of the matrix $D^{-1}B$, such that $|\lambda|\geq 1$. Then $\lambda D-B$ is a $K(m,r)$-diagonally dominant matrix, because for each $t_r \in e_r$ there exists an index $k\in\{1,\dots,m\}$ such that

$$E^{(k)}(\lambda D,t_r) = E^{(k)}(|\lambda| D,t_r) \geq E^{(k)}(D,t_r) > E^{(k)}(B,t_r).$$

$K(m,r)$ is a K-regular set, so that the matrix $\lambda D-B$ is a regular matrix. But, then the matrix $D^{-1}(\lambda D-B) = \lambda I-D^{-1}B$ is a regular matrix, which contradicts the assumption that $\lambda$ is an eigenvalue of $D^{-1}B$.

Hence, $\rho(D^{-1}B) < 1$, the matrix $I-D^{-1}B$ is regular and

$$(I-D^{-1}B)^{-1} = \sum_{i=0}^{\infty} (D^{-1}B)^{i} \geq 0.$$

Then $A^{-1}$ exists and $A^{-1} = (I-D^{-1}B)^{-1}D \geq 0$.

Analogously we can prove the following theorem.

*Theorem 2.* Let A be an L-matrix, whose diagonal elements are all positive. Let $K(m,r)$ be a $K_1$-regular set. If A is a $K(r(1),\dots,r(m))$- diagonally dominant matrix, then it is an M-matrix.

*Theorem 3.* Let A be an L-matrix, whose diagonal elements are all positive, such that at least one of the following conditions is satisfied:

(i) $a_{ii} > P_i(A)$, $i\in N$ (A is strictly diagonally dominant),

(ii) $a_{ii} > P_{i,\alpha}(A)$, $i\in N$, for some $\alpha\in[0,1]$,

(iii) $a_{ii} > P_i^{\alpha}(A)Q_i^{1-\alpha}(A)$, $i\in N$, for some $\alpha\in[0,1]$,

(iv) $a_{ii}a_{jj} > P_i(A)P_j(A)$, $i\in N$, $j\in N(i)$,

(v) $a_{ii}a_{jj} > P_i^{\alpha}(A)Q_i^{1-\alpha}P_j^{\alpha}(A)Q_j^{1-\alpha}(A)$, $i\in N$, $j\in N(i)$,

for some $\alpha\in[0,1]$,

(vi) For each $i\in N$ it holds that

$a_{ii} > P_i(A)$ or

$a_{ii} + \sum_{j\in J} a_{jj} > Q_i(A) + \sum_{j\in J} Q_j(A)$, where $J := \{i\in N : a_{ii} \le Q_i(A)\}$,

(vii) $a_{ii} > \min(P_i(A), Q_i^{*}(A))$, $i\in N$ and

$a_{ii}+a_{jj} > P_i(A)+P_j(A)$, $i\in N$, $j\in N(i)$,

(viii) $a_{ii} > Q_i^{(p)}(B)$, $i\in N$ and

$\sum_{j\in t_p} a_{ii} > \sum_{j\in t_p} P_i(A)$, $t_p\in e_p$, for some $p\in N$,

(ix) There exists $i\in N$ such that

$a_{ii}(a_{jj}-P_j(A)+|a_{ji}|) > P_i(A)|a_{ji}|$, $j\in N(i)$.

Then, A is an M-matrix.

P r o o f : (i) $a_{ii}=R_i(D) > R_i(B)=P_i(A)$, $i\in N$. Let $r=1$, $m=1$ and $e_1(x_1,x_2)=x_1, e_2=0$. Then the matrix A is $K(1,1)$-diagonally do-dominant and $K(1,1)=\{(e_1,e_2)\}$ is a regular set. So, from Theorem 1, it follows that A is an M- matrix.

Statements (ii)-(viii) can be proved similarly, by choosing

| case | m | r | s | $e_1^{(i)}, i=1,\dots,m$ | $e_2^{(i)}, i=1,\dots,m$ |
|---|---|---|---|---|---|
| (i) | 1 | 1 | n | $x_1$ | $\emptyset$ |
| (ii) | 1 | 1 | n | $\alpha x_1+(1-\alpha)x_2$ | $\emptyset$ |
| (iii) | 1 | 1 | n | $x_1^{\alpha}x_2^{1-\alpha}$ | $\emptyset$ |
| (iv) | 1 | 2 | n | $x_1x_2$ | $\emptyset$ |
| (v) | 1 | 2 | n | $(x_1x_2)^{\alpha}(x_3x_4)^{1-\alpha}$ | $\emptyset$ |
| (vi) | 2 | 1 | n | $x_1$ | $\emptyset$ |
| | | | | $x_2$ | $\sum_{j\in J} x_{j+k}$, k=card J |
| (vii) | 2 | 1 | 1 | $\min(x_1,x_2)$ | $\emptyset$ |
| | | 2 | n | $x_1+x_2$ | $\emptyset$ |
| (viii) | 2 | 1 | p | $x_2$ | $\emptyset$ |
| | | p | p | $\sum_{j\in t_p} x_j$ | $\emptyset$ |

(ix) can be proved in a similar way as in the proof of Theorem 1.

Note that strictly diagonally dominant matrices (SDD) satisfy all of the conditions (i)-(ix).

Statements (iii) and (iv) from Theorem 3 have been proved in [1].

## H-MATRIX CHARACTERIZATIONS

For any matrix $A = [a_{ij}] \in C^{n,n}$, we define $M(A) = [m_{ij}] \in R^{n,n}$ as follows

$$m_{ii} = |a_{ii}|,\ i \in N,\quad m_{ij} = -|a_{ij}|,\ i \in N,\ j \in N(i).$$

*Definition 7.* A matrix A is called an H-matrix iff M(A) is an M- matrix.

*Definition 8.* A matrix A is called generalized diagonally dominant (GDD) iff there exists a regular diagonal matrix W, so that AW is SDD.

*Theorem 4.* Let A be a matrix whose elements satisfy at least one of the conditions (i)-(ix) from Theorem 3, where all the diagonal elements of A are replaced by their modules. Then A is an H-matrix.

P r o o f : The matrix M(A) satisfies at least one of the conditions from Theorem 3 and it is an M-matrix.

Remark: Any irreducible diagonally dominant matrix is an H-matrix, too (see [2]).

*Theorem 5.* A matrix A is GDD if and only if it is an H-matrix.

P r o o f : Let A be GDD. Then, there exists a regular diagonal matrix W such that AW is SDD. Then, AW is an H- matrix, i.e. $M(AW) = M(A)M(W)$ is an M-matrix. Since $M(W)$ is regular and $M(W) > 0$, it follows that

$$(M(A))^{-1} = M(W)(M(AW))^{-1} \geq 0.$$

Conversely, if A is H-matrix, i.e. if M(A) is an M-matrix, then there exists a vector $z \in R^{n,n}$, $z > 0$, such that $M(A)z > 0$. It means that

$$|a_{ii}| z_i > \sum_{j \in N(i)} |a_{ij}| z_j \quad \text{for each } i \in N,$$

and we can choose the matrix $W = \mathrm{diag}(z_1, \ldots, z_n)$.

Similarly we can prove the theorems analogous to Theorems 1 and 2 on the characterizatons of H-matrices.

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REZIME

**NEKI REZULTATI O M- I H- MATRICAMA**

Matrica koja ima nepozitivne vandijagonalne elemente naziva se L-matrica, a regularna L-matrica, čija inverzna matrica ima nenegativne elemente, naziva se M-matrica. Ako se od proizvoljne matrice A napravi L-matrica M(A), čiji su elementi po modulu isti sa elementima matrice A, tada se matrica A naziva H-matrica ako i samo ako je M(A) M-matrica. U radu su date neke nove karakterizacije M- i H- matrica, koje sadrže i neke od ranije poznatih rezultata, kao posebne slučajeve.

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