

SEQUENTIAL CONTINUOUS MAPPINGS ON UNIFORM
SEMIGROUPS

Endre Pap

*University of Novi Sad, Faculty of Science,
Institute of Mathematics, Dr. I. Djuričića 4,
21000 Novi Sad, Yugoslavia*

ABSTRACT

An uniform boundedness theorem for the class of sequential continuous homomorphisms on uniform commutative semigroups is proved. For that purpose two kinds of boundedness are introduced. A diagonal type theorem on uniform semigroup is proved and by it a theorem on continuous convergence is obtained.

0. INTRODUCTION.

The paper deals with sequential continuous mappings defined on commutative uniform semigroup S_1 and with values in another commutative uniform semigroup S_2 .

The main purpose is to obtain an uniform boundedness type theorem - Theorem 2.1. By our knowledge this is the first uniform boundedness type theorem for homomorphisms on semigroups, i.e. on structures without scalar multiplication. For the purpose of that theorem, two kinds of boundedness (functionally and root) are introduced. We introduce also the subclass of root

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bounded subsets so-called K-root bounded subsets, in the same way as in [3].

We have proved also a diagonal type theorem, as a generalization of Lemma from [2], for uniform semigroups. This theorem enables us to obtain a theorem on continuous convergence.

1. BOUNDEDNESS ON AN UNIFORM SEMIGROUP

Let S be a uniform commutative semigroup with a neutral element. The uniformity of S is induced by a family D of pseudometrics d which satisfy the condition

$$d(x+x', y+y') \leq d(x, x') + d(y, y')$$

for $x, x', y, y' \in S$ (see [10], [8]). For a fixed $d \in D$ the functional f defined by $f(x) = d(x, 0)$, ($x \in S$) is a triangular functional (see [8], [9]), i.e. $f(0) = 0$, $f(x+y) \leq f(x) + f(y)$ and $f(x+y) \geq |f(x) - f(y)|$, ($x, y \in S$).

Let

$$F = \{f \mid f(x) = d(x, 0) \quad x \in S, \quad d \in D\}.$$

F is called the family of triangular functionals induced by the uniformity of S .

A sequence (x_n) from S converges to 0 iff $f(x_n) \rightarrow 0$ ($f \in F$). We shall introduce some kinds of boundedness on uniform semigroup S .

Definition 1.1. A subset A of S is functionally bounded if for each sequence (α_n) of nonnegative real numbers such that $\alpha_n \rightarrow 0$, $\alpha_n f(x) \rightarrow 0$ holds for each sequence (x_n) from A and each $f \in F$, where F is the family of triangular functionals induced by the uniformity of S .

Proposition 1.2. A subset A of S is functionally bounded iff $\frac{1}{n}f(x_n) \rightarrow 0$ as $n \rightarrow \infty$ for each sequence (x_n) from A

and each $f \in F$.

The preceding proposition obviously follows by the same property of non-negative real numbers.

Specially, a subset A of S is functionally bounded iff $\frac{1}{r_n} f(x_n) \rightarrow 0$ as $n \rightarrow \infty$ for each sequence (x_n) from A and each sequence (r_n) of natural numbers such that $r_n \rightarrow \infty$, as $n \rightarrow \infty$.

Let the algebraic semigroup S satisfy the condition

(R) For each $n \in \mathbb{N}$ and $x \in S$ there exists $y \in S$ such that $ny = x$. We denote by γ_n , the function $\gamma_n : S \rightarrow S$ such that $\gamma_n(x) = y$ (x and y from (R)).

Obviously, the map γ_n has the following properties

$$\gamma_n(x+y) = \gamma_n(x) + \gamma_n(y),$$

$$n[\gamma_n(x)] = x,$$

i.e. γ_n is a root function - [7].

The following definition holds for FLUSH convergence semigroup (see for the definitions [8]), which are more general than uniform semigroups.

Definition 1.3. A subset A of a FLUSH convergence semigroup S which satisfies the condition (R) is root bounded iff $\gamma_{r_n}(x_n) \rightarrow 0$ as $n \rightarrow \infty$ for each sequence (x_n) from S , and for each sequence (r_n) from \mathbb{N} such that $r_n \rightarrow \infty$.

We have the following characterization..

Proposition 1.4. A subset A of an uniform semigroup S which satisfies the condition (R) is root bounded iff holds

(i) For each neighborhood U of zero there exists $n \in \mathbb{N}$ such that

$$A \subset \{nu \mid u \in U\}.$$

Proof. Suppose that A is root bounded and that there exists a neighborhood U of zero such that for each $n \in \mathbb{N}$

$$A \not\subseteq \{nu \mid u \in U\}.$$

We choose a sequence (x_n) from A such that $x_n \in A \setminus \{nu \mid u \in U\}$ ($n \in \mathbb{N}$). By the construction $\gamma_n(x_n) \notin U$, ($n \in \mathbb{N}$). This implies $f(\gamma_n(x_n)) \neq 0$ for some $f \in F$. Contradiction to the assumption that A is root bounded.

Suppose now that for a subset A of S (i) holds but A is not root bounded. Then there exist a sequence (x_n) from A and $f \in F$ such that $f(\gamma_n(x_n)) \neq 0$. There exists a neighborhood U of zero such that $\gamma_n(x_n) \notin U$ for $n > n_0$ (we can take $n_0 = 0$). This implies $x_n \notin nU$, a contradiction.

Proposition 1.5. *Let S be a uniform semigroup which satisfies the condition (R). If a subset A of S is root bounded, then it is also functionally bounded.*

Proof. The assertion follows by the inequality

$$f(x_n) = f(n[\gamma_n(x_n)]) \leq nf(\gamma_n(x_n)),$$

where (x_n) is a sequence from A .

Proposition 1.6. *Let S be a locally convex space and A a subset of S . Then the following conditions are equivalent:*

- (i) A is functionally bounded;
- (ii) A is root bounded;
- (iii) A is bounded, i.e. for every neighborhood U of zero there exists $\varepsilon > 0$ such that $tA \subseteq U$ whenever $|t| < \varepsilon$.

For different type of boundedness on topological groups see [5], [6], [11] and on topological vector spaces - [4], [12].

Definition 1.6. Let S be a FLUSH convergence semigroup with zero element 0 and with the property (R). A sequence (x_n) in S is K -convergent sequence if each subsequence of (x_n) has a subsequence (x_{n_k}) such that

$$\sum_{k=1}^n x_{n_k} \rightarrow x \text{ for some } x \in S.$$

If S is a topological group then K -convergent sequence converges to 0 . The converse, in general, is false, but in complete spaces, it is true also the converse statement - [3], [8].

Definition 1.7. A subset A of a FLUSH convergence semigroup S which satisfies (R) is K -root bounded if it is root bounded and for each sequence (x_n) from A the sequence $(\gamma_n(x_n))$ is a K -convergent sequence.

Remark. If S is a topological group, then the supposition of root boundedness of A in the preceding definition is superfluous.

2. UNIFORM BOUNDEDNESS THEOREM

Theorem 2.1. Let A be a family of additive and sequentially continuous mappings from a commutative uniform semigroup S_1 , with property (R) to a commutative uniform semigroup S_2 . If the family A is pointwise functionally bounded, then it is uniformly functionally bounded on each K -root bounded subset of S_1 .

Proof. Let (T_n) be a sequence of mappings from A , let (x_n) be a sequence of elements from a K -root bounded subset A of S . We have to prove $\frac{1}{n}f(T_n(x_n)) \rightarrow 0$ as $n \rightarrow \infty$ for arbitrary $f \in F$.

There exists a sequence (r_n) of positive integers such that $r_n \rightarrow \infty$ and $\frac{1}{n}r_n \rightarrow 0$ as $n \rightarrow \infty$. For arbitrary $f \in F$ we

have

$$(1) \quad \frac{1}{n} f(T_n(x_n)) = \frac{1}{n} f(r_n T_n(\gamma_{r_n}(x_n))) \leq \frac{1}{n} r_n f(T_n(\gamma_{r_n}(x_n))).$$

Let

$$x_{ij} = \frac{1}{i} r_i f(T_i(\gamma_{r_j}(x_j))), \text{ for } i \neq j \text{ and } x_{ii} = 0 \text{ (} i \in \mathbb{N} \text{)}.$$

Since A is pointwise bounded we have

$$\lim_{i \rightarrow \infty} x_{ij} = 0 \quad (j \in \mathbb{N}).$$

Since A is root bounded we have $\gamma_{r_j}(x_j) \rightarrow 0$ as $j \rightarrow \infty$ and by continuity of T_i follows $\lim_{j \rightarrow \infty} x_{ij} = 0$, ($i \in \mathbb{N}$). Applying Antosik's Diagonal theorem from [1] we obtain an increasing sequence (p_i) of positive integers such that

$$(2) \quad \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0.$$

Since A is K -root bounded there exists a subsequence (s_i) of (p_i) such that

$$(3) \quad \sum_{j=1}^n \gamma_{s_j}(x_{s_j}) \rightarrow x$$

for some $x \in S_1$.

We have for arbitrary $p \in \mathbb{N}$

$$\begin{aligned} \frac{1}{s_i} r_{s_i} f(T_{s_i}(\gamma_{r_{s_i}}(x_{r_{s_i}}))) &\leq \sum_{\substack{j=1 \\ i \neq j}}^{i+p} \frac{1}{s_i} r_{s_i} f(T_{s_i}(\gamma_{r_{s_j}}(x_{r_{s_j}}))) + \\ &+ \left(\frac{1}{s_i} r_{s_i} f(T_{s_i}(\sum_{j=1}^{i+p} \gamma_{r_{s_j}}(x_{r_{s_j}}))), \quad (i \in \mathbb{N}). \right. \end{aligned}$$

For $p \rightarrow \infty$ we obtain by (3) and continuity of T_{s_i}

$$\frac{1}{s_i} r_{s_i} f(T_{s_i}(\gamma_{r_{s_i}}(x_{r_{s_i}}))) \leq \sum_{j=1}^{\infty} x_{s_i s_j} + \frac{1}{s_i} r_{s_i} f(T_{s_i}(x)).$$

Now, letting $i \rightarrow \infty$, we obtain by (2)

$$\frac{1}{s_i} r_{s_i} f(T_{s_i}(\gamma_{r_{s_i}}(x_{r_{s_i}}))) \rightarrow 0.$$

Hence by (1)

$$\frac{1}{s_i} f(T_{s_i}(x_{s_i})) \rightarrow 0.$$

The Urysohn property (U) of real numbers implies

$$\frac{1}{n} f(T_n(x_n)) \rightarrow 0.$$

3. CONTINUOUS CONVERGENCE

Diagonal Theorem 3.1. Let $[x_{ij}]$ ($i, j \in \mathbb{N}$) be a matrix of elements from a uniform semigroup S and let F be the induced family of triangular functionals. If for each increasing sequence (m_i) of positive integers there exists a subsequence of (m_i) such that

$$\begin{aligned} \text{(i)} \quad & \lim_{i \rightarrow \infty} f(x_{n_i n_j}) = 0 \quad (j \in \mathbb{N}, f \in F) \\ \text{and} \quad & \\ \text{(ii)} \quad & \lim_{i \rightarrow \infty} f\left(\sum_{j=1}^{\infty} x_{n_i n_j}\right) = 0, \quad (f \in F), \end{aligned}$$

where

$$f\left(\sum_{j=1}^{\infty} x_{n_i n_j}\right) := \lim_{s \rightarrow \infty} f\left(\sum_{j=1}^s x_{n_i n_j}\right), \quad (f \in F),$$

then

$$\lim_{i \rightarrow \infty} f(x_{ii}) = 0, \quad (f \in F).$$

Proof. The main idea of the proof is similar to the proof of Lemma from [2], so we give only a sketch of the proof.

Let f be a functional from the family F and let (n_i) be a subsequence of (m_i) such that (i) and (ii) hold. Then we

can choose a subsequence (p_i) of (n_i) such that

$$(1) \quad f(x_{p_i p_j}) < 2^{-i-j}, \quad (i, j \in \mathbb{N}, i \neq j).$$

By (ii) there exists a subsequence (q_i) of (p_i) such that

$$(2) \quad \lim_{i \rightarrow \infty} f\left(\sum_{j=1}^{\infty} x_{q_i q_j}\right) = 0.$$

Then (1) implies

$$(3) \quad \sum_{\substack{j=1 \\ j \neq i}}^{\infty} f(x_{q_i q_j}) < 2^{-i}.$$

The inequality

$$f(x_{q_i q_i}) \leq \sum_{\substack{j=1 \\ j \neq i}}^{i+p} f(x_{q_i q_j}) + f\left(\sum_{j=1}^{i+p} x_{q_i q_j}\right), \quad (i, p \in \mathbb{N}),$$

(2) and (3) imply $f(x_{q_i q_i}) \rightarrow 0$. Hence we obtain the assertion of Theorem.

Theorem 3.2. Let g_n , $n \in \mathbb{N}$, be additive and sequentially continuous mappings from a uniform semigroup S_1 to a uniform semigroup S_2 (endowed with the induced families F_1 and F_2 , respectively), both with neutral elements.

If

$$\lim_{n \rightarrow \infty} f(g_n(x)) = 0, \quad (x \in S, f \in F_2),$$

then

$$\lim_{n \rightarrow \infty} f(g_n(x_n)) = 0$$

for each K -sequence (x_n) from S_1 .

Proof. Put $x_{ij} = g_i(x_j)$, $(i, j \in \mathbb{N})$ and apply Diagonal theorem 3.1.

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REZIME

NIZOVNO NEPREKIDNA PRESLIKAVANJA NAD
UNIFORMNIM POLUGRUPAMA

U radu su uvedene dve vrste ograničenosti (funkcionalna i korena) nad uniformnim polugrupama. Ovo je omogućilo da se dokaže teorema o uniformnoj ograničenosti za nizovno neprekidne homomorfizme nad uniformnim polugrupama - Teorema 2.1.

Dokazana je dijagonalna teorema nad uniformnim polugrupama, te pomoću nje teorema 3.2. o neprekidnoj konvergenciji.

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