ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 17,1(1987)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 17,1(1987)

CONVERSES OF GENERALIZED BANACH CONTRACTION PRINCIPLES AND REMARKS ON MAPPINGS WITH A CONTRACTIVE ITERATE AT THE POINT

Bolesław Palczewski and Antoni Miczko
Politechnika Gdáńska, Wydzial Fizyki Technicznej i Matematyki Stosowanej, Majakowskiego 11/12, 80-952 Gdansk, Poland

ABSTRACT

The aim of this paper is to prove the converses of some generalizations of the Banach fixed point principles.

1. INTRODUCTION

In [2] and [13] some converses of the Banach fixed point principle are proved.

In this paper we give the converses of some generalized Banach fixed-point principles for families of nonnecessarily continuous mappings on a metric space (see Teorems 3.1-3.4).

In §4 we consider some mappings fulfilling Sehgal type conditions. We compare the mappings considered by F.Browder [3], O.Hadžić [10], O.Hadžić and Lj.Gajić [11],K.Iseki [12],

AMS Mathematics Subject Classification (1980): 47H10.

Key words and phrases: Converses of generalized Banach contraction principles, common fixed points, contractive iterate at the point, metric spaces.

J.Matkowski [15] and our paper [16] taking advantage, for this purpose, of the converses from §3.

The results of §3 of this paper are generalizations of the theorems of paper [20], which have been presented at the NATO Conference on "Nonlinear Functional Analysis and Fixed Point Theory", Maratea (Italy), April 22 - May 3, 1985.

2. NOTIONS, DEFINITIONS AND LEMMAS

Let X be a nonempty set on which two metrics d and e are given. We say that d is topologically equivalent to e, if the topologies τ_e and τ_d generated by e and d respectively are the same. It is obvious that d is topologically stronger than e iff $\tau_d \supset \tau_e$ or equivalent iff
$x_n \xrightarrow{\tau_d} x \Longrightarrow x_n \xrightarrow{\tau_e} x$.

Metrics d and e on X are C-equivalent, if every (x_n) is a Cauchy sequence in (X,d) iff it is a Cauchy sequence in (X,e).

REMARK 2.1. If d and e are metrics on X, then C-equivalence of d and e implies the topological equivalence of these metrics (see for example W.Opojcev [19]).

Let $F=(f_k)_{k\in N}$ be a countable family of selfmappings on a nonempty set X. We say that sequence $(x_n)_{n\in N_o}$ of the form

$$(2.1)\qquad \begin{cases} x_o \in X \\ x_n = f_n x_{n-1}, \quad n = 1,2,\ldots \end{cases}$$

is an (F,x_o)-orbit.

Let (X,d) be a metric space, F be a countable family of mappings $f_k: X \to X$, k=1,2,... and let $x_o \in X$ be given. The (F,x_o)-orbit (x_n) is a Cauchy (F,x_o)-orbit if (x_n) is a Cauchy sequence. We say that (X,d) is (F,x_o)-orbitally complete, if a Cauchy (F,x_o)-orbit is convergent to $x \in X$ and (X,d) is F-orbitally complete, if it is (F,x_o)-orbitally complete for any $x_o \in X$.

Mapping $g:X \to X$ is (F,x_o)-orbitally continuous at the point $x \in X$ if $d(gx_n, gx) \to 0$ for (F,x_o)-orbit (x_n) such that $d(x_n,x) \to 0$, as $n \to \infty$. Mapping g is F-orbitally continuous if it is (F,x_o)-orbitally continuous for every $x_o \in X$.

REMARK 2.2. If $F = \{f\}$ then the above definitions of the F-orbital completeness of (X,d) and the F-orbital continuity of g are slight modifications of the well-known corresponding definitions considered in Lj. Ćirić's papers [4], [5].

Let the families F and G of selfmappings on a metric space (X,d) be given and let the pair (F,G) have the properties:

(2.2) for each $f \in F$, there exists $g \in G$, that $fg = gf$,

(2.3) for each $f \in F$, $f(X) \subset Z$, where $Z = \bigcap_{g \in G} g(X) \neq \emptyset$,

(2.4) for each $\lambda \in (0,1)$, there exist a metric $\rho = \rho_\lambda$ on Z, and real numbers $\alpha, \beta \geq 0$, $\alpha + 2\beta = \lambda$, such that the inequality holds

$$\rho(f_1x, f_2y) \leq \alpha\rho(g_1x, g_2y) + \beta[\rho(g_1x, f_2y) + \rho(f_1x, g_2y)]$$

for $f_1 f_2 \in F$, $g_1 g_2 \in G$, $g_1f_1 = f_1g_1$, $g_2f_2 = f_2g_2$, $x \in g_1^{-1}(Z)$, $y \in g_2^{-1}(Z)$.

We can say that F is

1. a contraction (with respect to G) on (X,d), in abbreviation $F \in c_G(X,d)$, if (2.2)-(2.4) hold, where

a) $\beta = 0$ in condition (2.4)

b) condition (2.4) holds for arbitrary $f_1, f_2 \in F$, $f_1 = f_2 = f$ and $g_1 = g_2 = g \in G$, $fg = gf$,

c) ρ_λ is topologically equivalent to d on Z, and (Z, ρ_λ) is complete if (Z,d) is complete.

2. a quasi-contraction (with respect to G) on (X,d), to put it short $F \in qc_G(X,d)$, if

a) $\beta = 0$ in (2.4)

b) condition (2.4) holds for arbitrary $f_1 = f_2 = f \in F$ and $g_1 = g_2 = g \in G$, $fg = gf$,

c) ρ_λ is topologically stronger than d on Z and if (Z,d) is (h,x_o)-orbitally complete for some $x_o \in Z$ and some choice function $h: Z \to Z$, $h(x) \in f(g^{-1}(x))$, $x \in Z$, for any $f \in F$, $g \in G$, $fg = gf$, then (Z,ρ_λ) is (h,x_o)-orbitally complete.

REMARK 2.3. If in 1. (2. respectively), $G = \{id_X\}$ then we say that F is a contraction (quasi-contraction, respectively) on (X,d). In particular, if additionally $F = \{f\}$, then we get a contraction (a quasi-contraction, respectively) on (X,d) and then we write $f \in c(X,d)$ ($f \in qc(X,d)$, respectively).

3. a generalized contraction (with respect to G) on (X,d), briefly speaking $F \in gc_G(X,d)$, if

a) $\alpha = \beta = \frac{1}{3}\lambda$ in (2.4)

b) as b) in def. 1.

c) as c) in def. 1.

REMARK 2.4. If in 3., $G = \{id_X\}$ and $F = \{f\}$, then we say that f is a generalized contraction on (X,d) and thus we write $f \in gc(X,d)$.

The next two special classes are defined

4. $F \in (C.4)_G$ iff (2.2) - (2.4) hold, where

a) $\alpha = \beta = \frac{1}{3}\lambda$ in (2.4)

b) as c) in def. 1.

5. $F \in (C.5)_G$ iff (2.2) - (2.3) hold and (2.4) holds for $f_1,f_2 \in F^2 \setminus \Delta_F$ or $g_1,g_2 \in G^2 \setminus \Delta_G$, where

a) $\alpha = \beta = \frac{1}{3}\lambda$ in (2.4)

b) as c) in def. 1.,

Δ_F (Δ_G, resp.) denotes a diagonal in F^2 (in G^2, resp.).

REMARK 2.5. If $G = \{id_X\}$, $F = \{f_1,f_2\} \in (C.5)_G$, then the pair (f_1,f_2) fulfils the generalized contraction condition for pairs, and so we may write $(f_1,f_2) \in gcp(X,d)$.

We say that $a: R_+ \to R_+$ is a contractive gauge function (see W.Walter [25]), if it has the properties

(a_1) a is non-decreasing and continuous from the right

(a_2) $\lim_{n\to\infty} a^n(t) = 0$ for any $t > 0$.

The well-known Kwapisz's contractive gauge function (see for example [21], [26]) has the property (a_1) and in addition fulfils the following condition

(a_3) for any $q \in R_+$ there exists a maximal solution $m(q)$ of the equation $t = q + a(t)$, $t \in R_+$, which satisfies $m(0)=0$.

REMARK 2.6. It is obvious, that if $a: R_+ \to R_+$ fulfils (a_1) and (a_3) then a also has the property (a_2).

J.Matkowski [15], W.Walter [25], D.Xieping [26] and others consider the contractive gauge function $a: R_+ \to R_+$, which fulfils (a_1) and (a_2) and in addition has the property

(a_3') $\lim_{t\to\infty} (t - a(t)) = \infty$.

In paper [16] (see also [11]), we prove the following simple fact

LEMMA 2.1. ([16], Lemma 2.2) *Let* $Q = \{t \in R_+ : t \le q + a(t)\}$, $q \in R_+$, *where* a *fulfils* (a_1)-(a_2) *and* (a_3'). *Then*

(i) $Q \ne \emptyset$ *and* $\hat{a}(Q) \subset Q$, *where* $\hat{a}(t) = q + a(t)$, $t \ge 0$

(ii) Q *is bounded for each* $q > 0$, *the maximal solution* $m(q)$ *of the inequality* $t \le q + a(t)$ *is a fixed point of* $\hat{a}$ *and* $m(q) = \sup Q$

(iii) *the maximal solution* $m(0)$ *of the inequality* $t \le a(t)$ *is equal to* 0.

REMARK 2.7. The above lemma was proved without the assumption that a is continuous from the right. It is evident that if a fulfils (a_1)-(a_2) and (a_3'), then a fulfils (a_1) and (a_3).

Let (X,d) be a metric space and let $f:X \to X$. We define

$$O_f(x): = \{x, fx, \dots\},$$

$$O_f(x,p): = \{x, \dots , f^p x\},$$

$x \in X$, $p \in N$.

For the sequence $(x_n)_{n\in N_o}$, we define the following sets

$$O(x_n,p) = \{x_n, \dots , x_{n+p}\}$$

$$O(x_n,\infty) = \{x_n, x_{n+1}, \dots \},$$

$n \in N_o$, $p \in N$.

LEMMA 2.2. ([17], Lemma 1.1) *Let* $(x_n)_{n\in N_o}$ *be a sequence in a metric space* (X,d) *such that*

a) $d(x_o,x_1) \le q_o$

b) *the function* $a:R_+ \to R_+$ *fulfils* (a_1), *and there exists the maximal solution* $m(q_o)$ *of the inequality* $t \le q_o+a(t)$, $t \in R_+$, *and* $a^n(m(q_o) \to 0$ *as* $n \to \infty$, *where* $a^o = id_{R_+}$, $a^{n+1} = aa^n$, $n = 0,1,\dots,$

c) *for each* $n \in N_o$, $p \in N$, *the inequality holds*

$$\operatorname{diam}(O(x_n,p)) \le a(\operatorname{diam}(O(x_{n-1},p-1))).$$

Then $\operatorname{diam}(O(x_o,\infty)) \le m(q_o)$ *and* (x_n) *is a Cauchy sequence in* (X,d).

LEMMA 2.3. (Meyers Theorem [18]) *Let* X *be a metrizable space whose topology is generated by* d *and let* f *be continuous selfmapping on* X. *If there exists* $\bar{x} \in X$ *such that*

(2.5) $\bar{x} = f\bar{x}$

(2.6) $d(f^n x,\bar{x}) \to 0$ *as* $n \to \infty$ *for each* $x \in X$,

(2.7) *there exists an open neighbourhood* U *of* $\bar{x}$ *such that* $f^n(U) \to \{\bar{x}\}$, *i.e. for each neighbourhood* V *of* $\bar{x}$ *there exists* $n(V) \in N$, *that* $f^n(U) \subset V$ *for* $n > n(V)$, *then* $f \in c(X,d)$.

LEMMA 2.4. *Let* f *be a continuous selfmapping on a metric space* (X,d). *If there exists* $\bar{x} \in X$, *that*

(2.8) $d(\bar{x},fx) \leq a(d(\bar{x},x))$

for each $x \in X$, *where* $a:R_+ \to R_+$ *fulfils* $(a_1)-(a_2)$, *then* $f \in c(X,d)$.

PROOF. Obviously $\bar{x} = f\bar{x}$. From the inequality $d(f^n x,\bar{x}) \leq a^n(d(\bar{x},x))$, we get $d(f^n x,\bar{x}) \to 0$ as $n \to \infty$ for any $x \in X$.

Let $U = \{x \in X : d(\bar{x},x) < 1\}$. Then for each neighbourhood V of $\bar{x}$ there exists $n(V)$ that for $n > n(V)$, $f^n(U) \subseteq$ $\subseteq \{x \in X : d(\bar{x},x) < a^n(1)\} \subset V$, because $a^n(1) \downarrow 0$.

All the assumptions of Meyers Theorem hold and thus $f \in c(X,d)$.

REMARK 2.8. From Lemma 2.4, it follows that various continuous contractive selfmappings on a metric space (X,d) are contractions on (X,d) (see D.Xieping [26], Theorem 7 and some of our remarks from [21]).

3. CONVERSES OF GENERALIZED BANACH CONTRACTION PRINCIPLES FOR FAMILIES OF MAPPINGS

At first we shall prove the converse of generalized Banach fixed-point principle for a family of noncontinuous mappings.

THEOREM 2.1. *Let* (X,d) *be a metric space and let* F *be a family of selfmappings on* X. *Suppose that there exist the point* $\bar{x} \in X$ *and the contractive gauge function* $a:R_+ \to R_+$ *fulfilling* $(a_1)-(a_2)$ *so that the inequality holds*

(3.1) $d(\bar{x},fx) \leq a(d(\bar{x},x))$

for each $f \in F$ *and each* $x \in X$.
Then $F \in (C.4)_{\{id_X\}}$ *i.e. for each* $\lambda \in (0,1)$, *there exists a metric* d_λ, *topologically equivalent to* d, *and complete if* d *is complete such that the inequality* (3.2) *holds:*

(3.2) $d_\lambda(f_1x,f_2y) \leq \gamma(d_\lambda(x,y)+d_\lambda(x,f_2y)+d_\lambda(f_1x,y))$

for each $f_1,f_2 \in F$, $x,y \in X$, *where* $\gamma = \frac{1}{3}$.

PROOF. a) We define the family of balls $(B_n(r_o))_{n\in Z}$ as follows

$$B_n(r_o) = \{x \in X : d(\bar{x},x) \le \alpha_n\},$$

$n \in Z = \{0, \pm 1, \dots\}$, $r_o > 0$,
where

$$\alpha_n = \begin{cases} a^n(r_o) & \text{for } n = 0, 1, 2, \dots \\ \min\{r : r \in a^n(r_o)\} & \text{for } n = -1, -2, \dots, \end{cases}$$

where

$$a^{-p}(r_o) = \{r \in R_+ : a^p(r) = r_o\} = (a^p)^{-1}, \; p=1,2,\dots .$$

Function μ is defined in the following way

$$\mu(x,y) = \begin{cases} n(x) + n(y) & \text{for } x \ne \text{ and } y \ne \bar{x} \\ 2\min\{n(x), n(y)\} & \text{for } x = \bar{x} \text{ or } y = \bar{x}. \end{cases}$$

From the definition of μ we get the inequality for $f_1, f_2 \in F$, $x \ne \bar{x}$ and $y \ne \bar{x}$

$$\mu(f_1x,f_2y) \ge \max\{\mu(x,y),\mu(x,f_2y),\mu(f_1x,y)\} + 1.$$

For $\gamma = \frac{1}{3}\lambda$ we define

$$\rho_\gamma(x,y) = \begin{cases} 0 & \text{if } x = y = \bar{x} \\ \gamma^{\mu(x,y)}d(x,y) & \text{if } x,y \in X,\ x \ne \bar{x} \text{ or } y \ne \bar{x}. \end{cases}$$

If $x \ne \bar{x}$ and $y \ne \bar{x}$, then we can easily obtain the inequality $\rho_\gamma(f_1x,f_2y) \le \gamma(\rho_\gamma(x,y) + \rho_\gamma(x,f_2y) + \rho_\gamma(f_1x,y))$, $f_1f_2 \in F$.

However, if $x = \bar{x}$ and $y \ne \bar{x}$, then

$$\rho_\gamma(f_1\bar{x},f_2y) = \rho_\gamma(\bar{x},f_2y) =$$

$$= \gamma^{\mu(\bar{x},f_2y)}d(\bar{x},f_2y) \le \gamma^{\mu(\bar{x},y)}\gamma d(\bar{x},y) = \gamma\rho_\gamma(\bar{x},y)$$

and again we receive

$$\rho_\gamma(f_1\bar{x},f_2y) \le \gamma(\rho_\gamma(\bar{x},y) + \rho_\gamma(\bar{x},f_2y) + \rho_\gamma(f_1\bar{x},y)).$$

Thus for each $x,y \in X$ and $f_1,f_2 \in F$,

$$\rho_\gamma(f_1x,f_2y) \leq \gamma(\rho_\gamma(x,y) + \rho_\gamma(x,f_2y) + \rho_\gamma(f_1x,y)).$$

Obviously, $\rho_\gamma(x,y)=\rho_\gamma(y,x)$ and $\rho_\gamma(x,y)=0$ iff $x=y$, $x,y \in X$.

b) Now we shall introduce the functional for which the triangle inequality holds.

Let

$$d_\lambda(x,y) = \inf\{L_\gamma(\sigma_{xy}) : \sigma_{xy} \in \Sigma_{xy}\},$$

where Σ_{xy} denotes the set of chains $[x = x_o,\dots,x_m = y]$ and

$$L_\gamma(\sigma_{xy}) = \sum_1^m \rho_\gamma(x_i,x_{i-1}).$$

We have $d_\lambda(x,y) = d_\lambda(y,x)$, $d_\lambda(x,x) = 0$ and $d_\lambda(x,y) \leq$ $\leq d_\lambda(x,z) + d\ (z,y)$ for $x,y,z \in X$.

c) We shall prove that $d_\lambda(x,y) > 0$ for $x \neq y$, $x,y \in X$. Let $y \neq \bar{x}$ and let, for example, $n(x) \leq n(y)$ for some $y \in X$. Then:

$$d_\lambda(x,y) \geq \gamma^{2n(y)}\min\{d(x,y),d(x,B_{n(y)+1}(r_o)),d(y,B_{n(y)+1}(r_o))\}$$

and hence $d_\lambda(x,y) > 0$, where $d(x,A)$ denotes, as usual, the distance between point x and set A.

Analogically, if $y=\bar{x}$ then we have $d_\lambda(x,\bar{x}) \geq$ $\geq \gamma^{2n(x)}d(x,B_{n(x)+1}(r_o)) > 0$. Thus is this case we also have $d_\lambda(x,y) > 0$ for $x \neq y$, $x,y \in X$.

d) Metrics d_λ and d are topologically equivalent.

At first let $x \neq \bar{x}$ and let $x \in (B_{n(x)-k}(r_o))^o$ for some $k \in N$, and moreover $n(y) \geq n(x)$ for some $y \in X$, where for $A \subset X$, A^o denotes the d-interior of A.

We have the inequality

$$d_\lambda(x,y) \leq \gamma^{2\{n(x)-k\}}\min\{d(x,y),d(x,X\setminus(B_{n(x)-k}(r_o))^o)$$
$$d(y,X\setminus(B_{n(x)-k}(r_o))^o\}.$$

Let $\varepsilon > 0$. If $d(x,y) < \delta$, where

$$\delta = \varepsilon\gamma^{-2\{n(x)-k\}}\min\{1,d(x,X\setminus(B_{n(x)-k}(r_o))^o),$$
$$d(y,X\setminus(B_{n(x)-k}(r_o))^o)\}$$

then $d_\lambda(x,y) < \varepsilon$ and therefore, if $d(x_n,x) \to 0$, then $d_\lambda(x_n,x) \to 0$ as $n \to \infty$.

Let $n(y) \leq n(x)$ for some $x,y \in X$, $x \neq \bar{x}$. Then the inequality holds

$$d_\lambda(x,y) \geq \gamma^{2\{n(x)+k\}}\min\{d(x,y), d(x,B_{n(x)+k}(r_o)),$$
$$d(y,B_{n(x)+k}(r_o))\}$$

for some $k \in N$.
Let $0 < \varepsilon < \min\{d(x,B_{n(x)+k}(r_o)), d(y,B_{n(x)+k}(r_o))\}$. Hence, if $d_\lambda(x,y) < \delta = \varepsilon\delta^{2\{n(x)+k\}}$, then $d(x,y) < \varepsilon$. Thus $d_\lambda(x_n,x) \to 0$ implies $d(x_n,x) \to 0$ as $n \to \infty$.

Let now $x = \bar{x}$ and let, for example, $y \in B_o(r_o)$. We have

$$d_\lambda(\bar{x},y) \leq \rho_\gamma(\bar{x},y) \leq d(\bar{x},y)$$

and so if $d(x_n,\bar{x}) \to 0$ then $d_\lambda(x_n,\bar{x}) \to 0$ as $n \to \infty$.

For each $\varepsilon > 0$ there exists n_o such that $\min\{a^{n_o}(r_o)\} < \frac{\varepsilon}{2}$. If $d(\bar{x},y) > \varepsilon$ then $d(y,B_{n_o}(r_o)) > \frac{\varepsilon}{2}$ and $d_\lambda(\bar{x},y) \geq \gamma^{m_o} d(y,B_{n_o}(r_o)) > \gamma^{m_o}\frac{\varepsilon}{2}$ for some $m_o \in N$. If $d_\lambda(\bar{x},y) < \delta$, $\delta = \varepsilon\gamma^{-m_o}$, then $d(\bar{x},y) < \varepsilon$.

Therefore d_λ is topologically equivalent to d.

e) Let (X,d) be complete. We shall prove that in that case (X,d_λ) is also complete.

Let (x_m) be a Cauchy sequence in (X,d_λ) and let us assume that (x_m) is not convergent in (X,d_λ). Then we have $n(x_m) < p < \infty$ i.e. for each $m \geq 0$, $x_m \notin B_p(r_o)$.

Let $b = \alpha_p - \alpha_{p+1}$, where α_p is defined as in the part a) of this proof $\{p = 0, \pm 1, \ldots\}$. For sufficiently large n,

$$d(x_n, x_{n+j}) \leq b\gamma^{2(p+1)}.$$

It is easy to verify that

$$d_\lambda(x_n,x_{n+j}) \geq \gamma^{2(p+1)}\min\{d(x_n,x_{n+j}), b\}.$$

In that way

$$\gamma^{-2(p+1)} d_\lambda(x_n, x_{n+j}) \geq d(x_n, x_{n+j})$$

and (x_n) is a Cauchy sequence in (X,d). Then $d(x_n,x) \to 0$ for some $x \in X$ and from the topological equivalence of d and d_λ, $d_\lambda(x_n,x) \to 0$. This contradiction proves that (X,d_λ) is complete if (X,d) is complete.

Therefore $F \in (C.4)_{\{id_X\}}$ and the proof is complete.

In [21] we get the following characterization of continuous selfmappings on a metric space:

THEOREM 3.2.. ([21], Theorem 2.2.) *Let* (X,d) *be a metric space and let* $f_i: X \to X$, $i = 1, \ldots, n$, *be continuous mappings. Suppose that there exists the point* $\bar{x} \in X$ *and there is a real number* $\alpha > 0$ *so that the inequality holds*

$$d(\bar{x}, f_i x) \leq \alpha d(\bar{x}, x), \quad i = 1, \ldots, n, \tag{3.3}$$

for each $x \in X$. *Then the following conditions are equivalent*

(i) $\{f_1, \ldots, f_n\} \in c(X,d)$

(ii) $\{f_1, \ldots, f_n\} \in (C.5)_{\{id_X\}}$.

Now we shall formulate coincidence type converses for commuting selfmappings on metric spaces.

THEOREM 3.3.. *Let* F *and* G *be two families of selfmappings on a metric space* (X,d), *such that* (2.2)-(2.3) *hold. Assume that there exist* $x_o \in Z = \bigcap_{g \in G} g(X)$ *and a countable family of functions* $(h_i)_{i \in N}$, $h_i(x) \in f_i(g_i^{-1}(x))$, $x \in Z$, $f_i g_i = g_i f_i$, $i \in N$, *that* (Z,d) *is* $((h_i)_{i \in N}, x_o)$ *- orbitally complete. Then* $F \in (C.4)_G$ *iff there exist* $\bar{x} \in Z$ *and the contractive gauge function* a *fulfilling* (a_1)-(a_2) *such that the inequality holds*

$$d(\bar{x}, fx) \leq a(d(\bar{x}, gx)) \tag{3.4}$$

for each $f \in F$, $g \in G$, $fg = gf$, $x \in g^{-1}(Z)$.

PROOF 1. If $F \in (C.4)_G$, then by standard arguments we get that (F,x_o)-orbit (x_n), $x_n = h_n x_{n-1}$, $n = 1, 2, \ldots$, is a Cauchy orbit in (X,d_λ) and $d_\lambda(x_n,\bar{x}) \to 0$ for some $\bar{x} \in Z = \bigcap_{g\in G} g(X)$.

For each $f \in F$, $g \in G$, $fg = gf$, we have for $h:Z \to Z$, $h(x) \in f(g^{-1}(x))$, $x \in Z$, the inequality $d_\lambda(\bar{x},h\bar{x}) \leq$ $\leq d_\lambda(\bar{x},x_n) + d_\lambda(h_n x_{n-1},h\bar{x}) \leq d_\lambda(\bar{x},x_n) + \gamma[d_\lambda(x_{n-1},\bar{x}) + d_\lambda(x_{n-1},h\bar{x}) +$ $+ d_\lambda(x_n,\bar{x})]$ and taking $n \to \infty$, we get $d_\lambda(\bar{x},h\bar{x}) \leq \gamma d_\lambda(\bar{x},h\bar{x})$ i.e., $\bar{x} = h\bar{x}$.

Thus $\bar{x} \in f(g^{-1}(\bar{x}))$, and from $fg = gf$, $f\bar{x} = g\bar{x}$. But $d_\lambda(\bar{x},f\bar{x}) = d_\lambda(h_1\bar{x},f\bar{x}) = d_\lambda(f_1\bar{u},f\bar{x})$, where $\bar{u} \in g_1^{-1}(\bar{x})$, $g_1\bar{u} = \bar{x}$, $f_1 \in F$, $g_1 \in G$, $f_1g_1 = g_1f_1$.

Therefore

$$d_\lambda(\bar{x},f\bar{x}) \leq \gamma[d_\lambda(\bar{x},x) + d_\lambda(\bar{x},f\bar{x}) + d_\lambda(\bar{x},f\bar{x})] =$$

$$= 2\gamma d(\bar{x},f\bar{x}).$$

So $\bar{x} = f\bar{x} = g\bar{x}$.

If $\bar{y} = f\bar{y} = g\bar{y}$ for each $f \in F$, $g \in G$, $fg = gf$, then $d_\lambda(\bar{x},\bar{y}) =$ $= d_\lambda(f_1\bar{x},f_2\bar{y}) \leq 3\gamma d_\lambda(\bar{x},\bar{y})$ and $\bar{x} = \bar{y}$.

2. If the inequality $d(\bar{x},fx) \leq a(d(\bar{x},gx))$ holds for each $f \in G$, $g \in G$, $fg = gf$, then from Theorem 3.1, we get the searched for assertion.

THEOREM 3.4. ([21], Theorem 2.4.) *Let* (X,d) *be a metric space and let* $f_i,g_i : X \to X$, $f_ig_i = g_if_i$, $f_i(X) \subset Z$, $i = 1, \ldots, n$, $Z = \bigcap_{i=1}^{n} g_i(X)$. *Suppose that there exist the point* $\bar{x}$ *in* X *and a real number* $\alpha > 0$ *such that the inequality holds*

$$d(\bar{x},f_ix) \leq \alpha d(\bar{x},g_ix),\quad i = 1, \ldots, n, \tag{3.5}$$

for each $x \in X$. *If for each* $i \in \{1, \ldots, n\}$ *there exists a continuous choice function* $h_i : Z \to Z$, $h_i(x) \in f_i(g_i^{-1}(x))$, $x \in Z$, *then the following conditions are equivalent*

(i) $\{f_1,\ldots,f_n\} \in c_{\{g_1,\ldots,g_n\}}(X,d)$

(ii) $\{f_1,\ldots,f_n\} \in (C.5)_{\{g_1,\ldots,g_n\}}$.

REMARK 3.1. Theorems 3.1. and 3.3. are generalizations of the results of [22] (see [21], Theorems 2.1. and 2.3.). In [21] we prove converses of generalized Banach fixed-point theorems in the case where the contractive gauge function a in conditions (3.1) and (3.3) has the form $a(t) = \alpha t$, $t \in R_+$, $\alpha \in (0,1)$.

4. SOME REMARKS ON MAPPINGS WITH A CONTRACTIVE ITERATE AT THE POINT

Let (X,d) be a metric space. For mapping $f:X \to X$, the following conditions are taken into account

(4.1) (V.M.Sehgal [24]) there exists $\alpha \in [0,1)$, that for each $x \in X$ there exists $w(x) \in N$ that for any $y \in X$ the inequality holds

$$d(f^{w(x)}x, f^{w(x)}y) \leq \alpha d(x,y)$$

(4.2) (K.Iseki [12]) there exist α, β, $\gamma \geq 0$, $\alpha+4(\beta+\gamma)<1$, that for each $x \in X$ there is $w(x) \in N$ such that for any $y \in X$,

$$d(f^{w(x)}x, f^{w(x)}y) \leq \alpha d(x,y) + \beta[d(x,f^{w(x)}x) +$$

$$+ d(y,f^{w(x)}y)] + \gamma[d(f^{w(x)}x,f^{w(x)}y) + d(f^{w(x)}x,y)]$$

(4.3) (J.Matkowski [15]) there exists the function $\alpha:R_+^5 \to R_+$, nondecreasing with respect to each variable separately, that $a:R_+ \to R_+$, $a(t) = \alpha(t, t, 2t, t, 2t)$, $t \geq 0$, has properties (a_1)-(a_2) and (a_3'), and let for every $x \in X$ be a positive integer $w = w(x)$, such that for all $y \in X$, $d(f^w x, f^w y) \leq \alpha(d(x,y), d(x,f^w x), d(y,f^w y), d(x,f^w y), d(f^w x,y))$.

(4.4) (see [16]) there exists the function $\alpha:R_+^5 \to R_+$, nondecreasing related to each variable separately that $a:R_+ \to R_+$, $a(t) = \varphi(t, t, 2t, t, 2t)$, $t \in R_+$ is upper-semicontinuous and $r = 0$ is a unique solution of the inequality $t \leq a(t)$, $t \in R_+$, and for each $q > 0$ there exists a maximal solution $m(q)$ of the inequality $t \leq q + \varphi(t, q, 2t, t, q+t)$, $t \in R_+$.

Let $w:X \to N$ be such that for every $x,y \in X$,

$$d(f^{w(x)}x,f^{w(x)}y) \leq \varphi(d(x,y),d(x,f^{w(x)}x),d(y,f^{w(x)}y),$$

$$d(x,f^{w(x)}y),d(y,f^{w(x)}x)).$$

REMARK 4.1. V.M. Sehgal [24] and K.Iseki [12] assumed f to be a continuous mapping in a complete metric space. L.Guseman noted in [8] that the continuity condition of f in the fixed point theorems of Sehgal and Iseki is superfluous.

THEOREM 4.1. *Let* f *fulfil one of conditions* (3.1)-(3.4) *and let* (X,d) *be an* (F,x_o)*-orbitally complete metric space for some* $x_o \in X$, *where* $F(x) = f^{w(x)}x$, $x \in X$. *Then there exists a unique common fixed point* $\bar{x}$ *of* f *in* X *and* $d(f^n x,\bar{x}) \to 0$ *for each* $x \in X$. *If* f *fulfils* (4.1)-(4.4), *then* $f^m \in gc(X,d)$, $m = w(\bar{x})$. *If* f *is continuous in topology* τ_d *and if one of* (4.1)-(4.4) *holds, then* $f \in c(X,d)$.

PROOF. We have the following sequence of implications: (4.1) $\Rightarrow$ (4.2) $\Rightarrow$ (4.3) $\Rightarrow$ (4.4).

A) In [16] we prove (see also [21]) that if (4.4) holds, f has a unique fixed point $\bar{x}$ in X and $d(f^n x,\bar{x}) \to 0$ for each $x \in X$.

B) If f fulfils (3.4), then $d(\bar{x},f^m x) \le a(d(\bar{x},x))$, $x \in X$. Function a fulfils (a_1)-(a_2) and thus the assumptions of Theorem 3.1. are fulfilled. In consequence $f \in gc(X,d)$.

C) If f fulfils (4.1)-(4.4) and f is a continuous mapping in the topology generated by d, then all the assumptions of Meyers Theorem [18] hold for the iterate f^m of f, $m = w(\bar{x})$.

For example, there exists an open neighbourhood U of $\bar{x}$ that $f^{nm}(U) \to \{\bar{x}\}$. But in that case also $f^n(W) \to \{\bar{x}\}$, where

$$W = \bigcap_{j=0}^{m-1} f^{-j}(U)$$

(compare P.Meyers [18]). As a result $f \in c(X,d)$.

(4.5) (W.Walter [25], (C.5)) for every $x \in X$ there exists a positive integer $w(x)$ such that for $n \ge w(x)$ and $y \in X$,

$$d(f^n x,f^n y) \le a(\mathrm{diam}(O_f(x,n) \cup O_f(y,n))),$$

where $a: R_+ \to R_+$ fulfils (a_1)-(a_2) and (a_3')

(4.6) for every $x \in X$ there exists $w(x) \in N$ such that for $n \geq w(x)$, $y \in X$,

$$d(f^n x, f^n y) \leq a(\operatorname{diam}(O_f(x,n) \cup O_f(x,n))),$$

where $a: R_+ \to R_+$ fulfils (a_1) and (a_3).

THEOREM 4.2. *Let* (X,d) *be a complete metric space and* f *be a continuous selfmapping on* X *fulfilling one of the conditions* (4.5)-(4.6). *Then there exists a unique fixed point* $\bar{x}$ *of* f *in* X *and* $d(f^n x, \bar{x}) \to 0$ *for each* $x \in X$. *Moreover,* $f \in qc(X,d)$.

PROOF. A) We have the implication (4.5)$\Rightarrow$(4.6). It is easy to verify, on the basis of Lemma 2.2, that $\operatorname{diam}(O_f(x)) < \infty$ for each $x \in X$. Let $\rho(x,x) = 0$ and $\rho(x,y) = \operatorname{diam}(O_f(x) \cup O_f(y))$ for $x,y \in X$. It is obvious that (X,ρ) is a metric space and for each $x \in X$ there exists $w(x) \in N$, that for any $y \in X$, $\rho(f^{w(x)}x, f^{w(x)}y) \leq a(\rho(x,y))$, $x,y \in X$. We also have $\rho(x,y) \geq d(x,y)$ for each $x,y \in X$.

B) Now we shall prove that (X,ρ) is F-orbitally complete, where $Fx = f^{w(x)}x$, $x \in X$.

Let (x_n) be (F,x_o)-orbit for some x_o, i.e. $x_n = f^{w(x_{n-1})}x_{n-1}$, $n = 1,2,\ldots$.

As in the proof of Theorem 2.2. of [21], we have

$$\sup\{\sup\{\rho(x_k, f^l x_k) : l \geq 1\} : k \geq n\} \to 0 \quad \text{as} \quad n \to \infty$$

and so (x_n) is a Cauchy sequence in (X,d) and $d(x_n, \bar{x}) \to 0$ for some $\bar{x} \in X$.

Moreover, $\sup\{\sup\{d(x_k, f^l x_k) : l \geq 0\} : k \geq n\} \to 0$ and thus $\rho(x_n, f^k x_n) \to 0$ as $n \to \infty$ for each $k \in N$.

We have

$$\rho(\bar{x},x_n) = \operatorname{diam}(O_f(\bar{x}) \cup O_f(x_n)) =$$

$$\max\{\sup\{d(f^k\bar{x},f^l\bar{x}):k,l \geq 0\}, \sup\{d(f^kx_n,f^lx_n):k,l\geq 0\},$$

$$\sup\{d(f^k\bar{x},f^lx_n):k,l \geq 0\}\}.$$

On the other hand, we get estimations

$$d(f^k\bar{x},f^l\bar{x}) \leq d(f^k\bar{x},f^kx_n) + d(f^kx_n,x_n) + d(x_n,f^lx_n) + \\ + d(f^lx_n,f^l\bar{x}),$$

$$d(f^kx_n,f^lx_n) \leq d(f^kx_n,x_n) + d(x_n,f^kx_n) \quad \text{and}$$

$$d(f^k\bar{x},f^lx_n) \leq d(f^k\bar{x},f^kx_n) + d(f^kx_n,x_n) + d(x_n,f^lx_n)$$

and from the continuity of f in τ_d, we obtain $\rho(x_n,\bar{x}) \to 0$ as $n \to \infty$.

From Theorem 4.1, we get the assertion

(4.7) (O.Hadžić [9], Theorem 2) Let $f,g_1,g_2:X \to X$ be such that

(i) f,g_1,g_2 are continuous in τ_d

(ii) $f(X) \subset Z$, $Z = g_1(X) \cap g_2(X)$

(iii) $fg_i = g_if$, $i = 1,2$.

Let there exist $w:X \to N$ and $q \in [0,1)$ that

(iv) $d(f^{w(x)}x,f^{w(x)}y) \leq q \min\{d(g_1x,g_2y), d(g_2x,g_1y)\}$,

$x,y \in X$,

(v) for every $x \in X$, one of the sets $\{f^mg_1^px : p \in N,\ m \in \{0,1,\ldots,w(x)-1\}\}$ and $\{f^mg_2^px : p \in N,\ m \in \{0,1,\ldots,w(x)-1\}\}$

is bounded.

THEOREM 4.3. (compare Theorem 4.1. of [21]) *Let* (X,d) *be a complete metric space and let* $f,g_1,g_2:X \to X$ *fulfil* *condition* (4.7). *Then for each* $\lambda \in (0,1)$, *there exists a metric* d_λ *on* Z, *topologically equivalent to* d, *and complete if* d *is complete, such that* $d_\lambda(f^m x, f^m y) \leq \gamma(d_\lambda(g_i x, g_j y) + d_\lambda(g_i x, f^m y) + d_\lambda(f^m x, g_j y))$ *for* $x \in g_1^{-1}(Z)$, $y \in g_j^{-1}(Z)$, $i,j \in \{1,2\}$, $\gamma = \frac{1}{3}\lambda$, $m = w(\bar{x})$, $\bar{x} = f\bar{x} = g_1\bar{x} = g_2\bar{x}$.

If in addition there exist continuous choice functions $h_i : Z \to Z$, $h_i(x) \in f^m(g_i^{-1}(x))$, $x \in X$, $i = 1,2$, $m = w(\bar{x})$, *then for each* $\alpha \in (0,1)$ *there exists a metric* d_α, *topologically equivalent to* d, *and complete if* d *is complete, such that* $d_\alpha(f^m x, f^m y) \leqslant \alpha d_\alpha(g_i x, g_i y)$, $x,y \in g_i^{-1}(Z)$, $i = 1,2$, $m = w(\bar{x})$.

(4.8) (O.Hadžić, Lj.Gajić [10], Theorem 1) Let f, g_1, g_2 be such selfmappings on (X,d), that (4.7)(i)-(iii) holds. Suppose that there exist $w : X \in N$ and nondecreasing $q : [0,\infty) \to [0,1)$, $\lim_{t\to\infty} t(1 - q(t)) = \infty$, such that

(vi) $$d(f^{w(x)}x, f^{w(x)}y) \leq$$
$$\leq \min\{q(d(g_1 x, g_2 y)) \cdot d(g_1 x, g_2 y),\ q(d(g_2 x, g_1 y)) \cdot d(g_2 x, g_1 y)\}, \quad x,y \in X,$$

(vii) for some $x_o \in X$ one of the sets

$$\{f^m g_1^p x_o : p \in N,\ m \in \{0,\dots,w(x_o)-1\}\} \text{ and}$$

$$\{f^m g_2^p x_o : p \in N,\ m \in \{0,\dots,w(x_o)-1\}\}$$

is bounded.

THEOREM 4.4. *Let* (X,d) *be a complete metric space and let* $f, g_1, g_2 : X \to X$ *fulfil condition* (4.8). *If function* $q : [0,\infty) \to [0,1)$ *is continuous from the right, then* $f^m \in (C.4)_{\{g_1,g_2\}}$, $m = w(\bar{x})$, $\bar{x} = f\bar{x} = g_1\bar{x} = g_2\bar{x}$. *If, besides that, there exist continuous choice functions* $h_i : Z \to Z$, $h_i(x) \in f^m(g_i^{-1}(x))$, $x \in Z$, $i = 1,2$, $m = w(\bar{x})$, *then* $f \in c_{\{g_1,g_2\}}(X,d)$.

PROOF. From Theorem 1 of [10], there exists a unique common fixed point $\bar{x}$ of f, g_1, g_2 in X, $\bar{x} \in Z$.

We have the inequality

$$d(\bar{x},f^m x) \le \min\{q(d(\bar{x},g_2x))d(\bar{x},g_2x),\ q(d(\bar{x},g_1x))d(\bar{x},g_1x)\}$$

i.e. $d(\bar{x},f_2^m x) \le q(d(\bar{x},g_ix))d(\bar{x},g_ix)$ for each $x \in X$, $i = 1,2$. Thus the assumptions of Theorem 3.3. are fulfilled and $f \in (C.4)_{\{g_1,g_2\}}$.

If there exist continuous choice functions $h_1,h_2:Z \to Z$, all the assumptions of Theorem 3.4. hold and $f \in c_{\{g_1,g_2\}}(X,d)$ (iff $f \in (C.5)_{\{g_1,g_2\}}$). The proof is complete.

(4.9) (O.Hadžić, L.Gajić [10], Theorem 2). Let $G=\{g_1,g_2\}$, where $g_1,g_2:X \to X$ are continuous mappings. Let F be a countable family of mappings $f_i:X \to Z$, $i = 1,2,\ldots$, $Z = g_1(X) \cap g_2(X)$ such that $f_ig_j = g_jf_i$, $i \in N$, $j \in \{1,2\}$. Suppose that $q:[0,\infty) \to [0,1)$ is a nondecreasing continuous function and for every $x,y \in X$:

$$d(f_ix,f_jy) \le q(d(g_1x,g_2y))$$

$i \neq j$, $i,j \in N$.

THEOREM 4.5. *Let* (X,d) *be a complete metric space and let* F *and* G *fulfil* (4.9). *Then* $F \in (C.4)_G$. *If card* $F = n$, *i.e.,* $F = \{f_1,\ldots,f_n\}$ *and for each* $i \in \{1,\ldots,n\}$ *there exist continuous choice functions* $h_i:Z \to Z$, $h_i(x) \in f_i(g_k^{-1}(x))$, $x \in Z$, $k = 1$ *and* $k = 2$, *then* $F \in (C.5)_G$.

PROOF. O.Hadžić and L.Gajić in [10] proved that there exists a unique common fixed point $\bar{x}$ of f_i, $i = 1,\ldots$ and g_1 and g_2 in X, $\bar{x} \in Z$. Thus we have the inequality

$$d(\bar{x},f_jx) \le q(d(\bar{x},g_kx))d(\bar{x},g_kx)$$

for $j \in N$ and $k \in \{1,2\}$. Hence from Theorem 2.3, $F \in (C.4)_G$.

The second part of the assertion follows from Theorem 2.4.

FINAL REMARK. Theorems of paragraph 3 of this work give a new possibility of presenting a wide range of contractive type mappings. An attempt of a survey of mappings belonging to classes c(X,d), gc(X,d) and qc(X,d) has already been made by us before in [21].

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REZIME

INVERZIJE UOPŠTENJA BANAHOVOG PRINCIPA
KONTRAKCIJE I PRIMEDBE O PRESLIKAVANJIMA
SA KONTRAKTIVNOM ITERACIJOM U TAČKI

U ovom radu su dokazane inverzije nekih uopštenja Banahovog principa kontrakcije.

Received by the editors August 11, 1985.