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NEUTRICES AND THE CONVOLUTION OF DISTRIBUTIONS

Brian Fisher

*Department of Mathematics, The University,
Leicester, LE1 7RH, England*

ABSTRACT

A new definition of the convolution of distributions is given and some of its properties are investigated.

☆

The convolution of two functions is normally defined as follows, see for example Sikorski [5].

Definition 1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exists.

It follows easily from the definition that if $(f * g)(x)$ exists

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then $(g * f)(x)$ exists and

$$(f * g)(x) = (g * f)(x) \tag{1}$$

and if $(f * g)'(x)$ and $(f * g')(x)$ (or $(f' * g)(x)$) exist, then

$$(f * g)'(x) = (f * g')(x) \text{ (or } (f' * g)(x)). \tag{2}$$

The following theorem also holds and it is an immediate consequence of Hölder's inequality for integrals.

Theorem 1. Let f and g be functions in $L^p(-\infty, \infty)$ and $L^q(-\infty, \infty)$ respectively, where $1/p + 1/q = 1$. Then the convolution $(f * g)(x)$ exists for all values of x.

Now suppose that the convolution $(f * g)(x)$ exists for all values of x and let ϕ be an arbitrary test function in the space K of infinitely differentiable functions with compact support. Then

$$\begin{aligned}((f * g)(x), \phi(x)) &= \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\infty} f(t)g(x-t)\,dt\,dx \\ &= \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(t)\phi(y+t)\,dt\,dy\end{aligned}$$

which for convenience we will write as

$$((f * g)(x), \phi(x)) = (g(y), (f(x), \phi(x+y))),$$

even though the infinitely differentiable function $(f(x), \phi(x+y))$ does not necessarily have bounded support.

This leads us to the following definition for the convolution of certain distributions f and g, see for example Gelfand and Shilov [3].

Definition 2. Let f and g be distributions satisfying either of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the convolution $f * g$ is defined by

$$((f * g)(x), \phi(x)) = (g(y),(f(x), \phi(x + y)))$$

for arbitrary test function ϕ in K.

Note that with this definition, if f has bounded support, then $(f(x), \phi(x + y))$ is in K and so $(g(y),(f(x),\phi(x + y)))$ is meaningful. If on the other hand either g has bounded support or the supports of f and g are bounded on the same side, then the intersection of the supports of $g(y)$ and $(f(x), \phi(x + y))$ is bounded and so $(g(y),(f(x), \phi(x + y)))$ is again meaningful.

It follows that if the convolution $f * g$ exists by this definition then equations (1) and (2) always hold.

Definitions 1 and 2 are very restrictive and can only be used for a small class of distributions. In order to extend the convolution to a larger class of distributions, Jones [4] gave the following definition.

Definition 3. Let f and g be distributions and let τ be an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leqslant \tau(x) \leqslant 1$,

(iii) $\tau(x) = 1$ for $|x| \leqslant \frac{1}{2}$,

(iv) $\tau(x) = 0$ for $|x| \geqslant 1$.

Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1,2, \ldots$. Then the convolution $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$, providing the limit h exists in the sense that

$$\lim_{n\to\infty} (f_n * g_n , \phi) = (h , \phi)$$

for all test functions ϕ in K.

Note that in this definition the convolution $f_n * g_n$ exists by definition 2 since f_n and g_n both have bounded supports. It is also clear that if the limit of the sequence $\{f_n * g_n\}$ exists, so that the convolution $f * g$ exists, then equation (1) holds. However equations (2) need not necessarily hold since Jones proved that

$$1 * \operatorname{sgn} x = x = \operatorname{sgn} x * 1$$

and

$$(1 * \operatorname{sgn} x)' = 1, \quad 1' * \operatorname{sgn} x = 0, \quad 1 * (\operatorname{sgn} x)' = 2.$$

An alternative extension of definitions 1 and 2 was given in [2]. To distinguish this definition from definition 3 the convolution of two distributions f and g was denoted by $f \circledast g$ when it existed.

Definition 4. Let f and g be distributions and let f_n be defined as in definition 3. Then the convolution $f \circledast g$ is defined as the limit of the sequence $\{f_n * g_n\}$, providing the limit h exists in the sense that

$$\lim_{n\to\infty} (f_n * g , \phi) = (h , \phi)$$

for all test functions ϕ in K.

In this definition the convolution $f_n * g$ is in the sense of definition 2, the distribution f_n having bounded support. We also note that because of the lack of symmetry in this definition the convolution of two distributions is not always commutative.

In the following we are now going to give another non-commutative extension of definitions 1 and 2. This definition is also possibly an extension of definition 4 since not only are all the results proved in [2] in agreement with the new definition but further convolutions exist which are not defined by definition 4. Whether or not there exist distributions f and g which give different results for the convolution $f \circledast g$, or for which the convolution $f \circledast g$ exists by definition 4 but not by the new definition, are open questions.

First of all we need the following definition given by van der Corput [1].

Definition 5. A neutrix N is a commutative additive group of functions $\nu : N' \to N''$ (where the domain N' is a set and the range N'' is a commutative additive group) with the property that if ν is in N and $\nu(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The functions in N are said to be negligible. Now suppose that N' is contained in a topological space with a limit point b which is not in N' and let N be a commutative additive group of functions $\nu : N' \to N''$ with the property that if N contains a function of ξ which tends to a finite limit γ as ξ tends to b, then $\gamma = 0$. It follows that N is a neutrix. If now $f : N' \to N''$ and there exists a constant β such that $f(\xi) - \beta$ is negligible in N, then β is called the neutrix limit of $f(\xi)$ as ξ tends to b and we write

$$\operatorname*{N-lim}_{\xi \to b} f(\xi) = \beta,$$

where β is always unique if it exists.

Definition 6. Let f and g be distributions and let τ_n be the infinitely differentiable function defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

where τ is defined as in definition 3. Let

$$f_n(x) = f(x)\tau_n(x)$$

for $n = 1,2, \ldots$. Then the convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, providing the limit h exists in the sense that

$$\underset{n\to\infty}{\text{N-lim}}\, (f_n * g, \phi) = (h, \phi)$$

for all test functions ϕ in K, where N is the neutrix having domain $N' = \{1,2, \ldots ,n, \ldots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1,2, \ldots$, and all functions $f(n)$ for which $\lim_{n\to\infty} f(n) = 0$.

The convolution $f_n * g$ in this definition is again in the sense of definition 2, the distribution f_n having bounded support since the support of τ_n is contained in the interval $(-n - n^{-n}, n + n^{-n})$.

From now on all the convolutions denoted by $f * g$ will be as defined in definitions 1 or 2 and those denoted by $f \circledast g$ will be as defined in definition 6.

Theorem 2. Let f and g be functions in $L^p(-\infty,\infty)$ and $L^q(-\infty,\infty)$ respectively, where $1/p + 1/q = 1$. Then the convolution $f \circledast g$ exists and

$$f \circledast g = f * g.$$

Proof. For arbitrary $\epsilon > 0$ we have

$$|f * g - f_n * g| = \left| \int_{-\infty}^{\infty} f(t)g(x-t)[1 - \tau_n(t)]\,dt \right|$$

$$\leqslant \int_{|t| \geqslant n} |f(t)g(x-t)|\,dt < \epsilon$$

for all n greater than some n_0. Thus if ϕ is an arbitrary test function in K then

$$|(f * g, \phi) - (f_n * g, \phi)| \leqslant \sup\{|\phi(x)|\}\epsilon$$

for $n > n_0$ and it follows that

$$\lim_{n\to\infty} (f_n * g, \phi) = (f * g, \phi) = \underset{n\to\infty}{\text{N-lim}}\,(f_n * g, \phi),$$

or equivalently that

$$f \circledast g = f * g.$$

This completes the proof of the theorem.

This theorem therefore shows that definition 6 is an extension of definition 1. The next theorem shows that definition 6 is also an extension of definition 2.

Theorem 3. Let f and g be distributions satisfying either condition (a) or condition (b) of definition 2. Then the convolution $f \circledast g$ exists and

$$f \circledast g = f * g.$$

Proof. Suppose first of all that the support of f is bounded. Then $f = f_n$ for large enough n and so

$$\lim_{n\to\infty} (f_n * g, \phi) = (f * g, \phi) = \underset{n\to\infty}{\text{N-lim}} (f_n * g, \phi)$$

for all test functions ϕ in K. The result follows in this case.

Now suppose that the support of g is contained in the bounded interval (a,b) and let ϕ be an arbitrary test function in K with support contained in the bounded interval (c,d). Then

$$(f * g - f_n * g, \phi) = (g(y),(f(x) - f_n(x), \phi(x+y)))$$

$$= \int_a^b g(y) \int_{c-y}^{d-y} f(x)[1 - \tau_n(x)]\phi(x+y)\,dx\,dy$$

$$= 0$$

for large enough n and the result follows in this second case.

Finally suppose that the supports of f and g are bounded on the same side, say on the left, so that the supports of f and g are contained in the half-bounded intervals (a,∞) and (b,∞) respectively. Then if ϕ is an arbitrary test function in K with its support contained in the bounded interval (c,d) we have

$$(f * g - f_n * g, \phi) = \int_b^\infty g(y) \int_{c-y}^{d-y} f(x)[1 - \tau_n(x)]\phi(x+y)\,dx\,dy.$$

Now $f(x) = 0$ if $x < a$ and so

$$\int_{c-y}^{d-y} f(x)[1 - \tau_n(x)]\phi(x + y)\,dx = 0$$

if $y > d - a$. Thus

$$(f * g - f_n * g, \phi) = \int_b^{d-a} g(y)\int_{c-y}^{d-y} f(x)[1-\tau_n(x)]\phi(x+y)\,dx\,dy$$

$$= 0$$

for large enough n and the result follows in this third case. This completes the proof of the theorem.

Having proved that definition 6 is an extension of both definitions 1 and 2, we will now consider a particular convolution which is defined by definition 6 but which is not defined by definitions 1, 2, 3 or 4.

Example. $\quad x^2 \circledast (x^2+\epsilon^2)^{-1} = \frac{\pi}{\epsilon}(x^2-\epsilon^2). \qquad (3)$

We put

$$(x^2)_n = x^2\tau_n(x), \quad f_\epsilon(x) = (x^2+\epsilon^2)^{-1}.$$

Then the convolution $(x^2)_n * f_\epsilon(x)$ exists by definition 1 and

$$(x^2)_n * f_\epsilon(x) = \int_{-\infty}^{\infty} \frac{(x-t)^2\tau_n(x-t)}{t^2+\epsilon^2}\,dt$$

$$= \int_{-n+x}^{n+x} \frac{(x-t)^2}{t^2+\epsilon^2}\,dt + \int_{n+x}^{n+n^{-n}+x} \frac{(x-t)^2\tau(x-t)}{t^2+\epsilon^2}\,dt +$$

$$+ \int_{-n-n^{-n}+x}^{-n+x} \frac{(x-t)^2\tau(x-t)}{t^2+\epsilon^2}\,dt.$$

Now

$$\int_{-n+x}^{n+x} \frac{(x-t)^2}{t^2+\epsilon^2}\,dt = \int_{-n+x}^{n+x}\left[\frac{x^2-\epsilon^2}{t^2+\epsilon^2} - \frac{2xt}{t^2+\epsilon^2} + 1\right]dt$$

$$= \frac{x^2-\epsilon^2}{\epsilon}\left[\tan^{-1}\frac{n+x}{\epsilon} - \tan^{-1}\frac{x-n}{\epsilon}\right] - x\ln\frac{(n+x)^2+\epsilon^2}{(n-x)^2+\epsilon^2} + 2n,$$

$$\left| \int_{n+x}^{n+n^{-n}+x} \frac{(x-t)^2 r(x-t)}{t^2+\epsilon^2}\,dt \right| \leqslant \frac{(n+n^{-n})^2 n^{-n}}{(n+x)^2+\epsilon^2} = O(n^{-n})$$

and similarly

$$\int_{-n-n^{-n}+x}^{-n+x} \frac{(x-t)^2 r(x-t)}{t^2+\epsilon^2}\,dt = O(n^{-n}).$$

It follows that

$$\underset{n\to\infty}{\text{N-lim}} \left((x^2)_n * f_\epsilon(x), \phi(x)\right) = \frac{\pi}{\epsilon}(x^2-\epsilon^2, \phi)$$

for arbitrary test function ϕ in K and equation (3) follows.

We now put

$$\delta_\epsilon(x) = \frac{\epsilon}{\pi} f_\epsilon(x) = \frac{\epsilon}{\pi(x^2+\epsilon^2)}.$$

It is well-known that

$$\lim_{\epsilon\to 0} \delta_\epsilon(x) = \delta(x),$$

where δ is the Dirac delta-function, and it follows from equation (3) that

$$x^2 \circledast \delta_\epsilon = x^2 - \epsilon^2.$$

Thus

$$\lim_{\epsilon\to 0} x^2 \circledast \delta_\epsilon = x^2 = x^2 * \delta,$$

a result we should certainly hope to obtain since δ acts as the identity with the convolution product.

It can be proved similarly that

$$(x^2+\epsilon^2)^{-1} \circledast x^2 = \frac{\pi}{\epsilon}(x^2-\epsilon^2)$$

and so we also have

$$\lim_{\epsilon\to 0} \delta_\epsilon \circledast x^2 = x^2 = \delta * x^2.$$

We now prove some more general results.

Theorem 4. Let f and g be distributions and suppose that the convolution $f \circledast g$ exists. Then the convolution $f \circledast g'$ exists and

$$(f \circledast g)' = f \circledast g'.$$

Proof. Since the convolution $f_n * g$ exists by definition 2 we have

$$(f_n * g)' = f_n * g'.$$

Thus

$$((f \circledast g)', \phi) = -(f \circledast g, \phi') = -\underset{n\to\infty}{\text{N-lim}}\,(f_n * g, \phi')$$

$$= \underset{n\to\infty}{\text{N-lim}}\,((f_n * g)', \phi) = \underset{n\to\infty}{\text{N-lim}}\,(f_n * g', \phi)$$

for arbitrary test function ϕ in K. It follows that the convolution $f \circledast g'$ exists and

$$(f \circledast g)' = f \circledast g'.$$

This completes the proof of the theorem.

This result also holds for the convolution given by definition 4, see theorem 3 of [2]. However, this result does not hold in general for the convolution given by definition 3, see [4].

Theorem 5. Let f be an odd distribution. Then the convolution $f \circledast 1$ exists and

$$f \circledast 1 = 0.$$

Proof. Since the convolution $f_n * 1$ exists by definition 2, equation (1) holds and so

$$(f_n * 1, \phi) = (1 * f_n, \phi)$$

$$= (f_n(y), (1, \phi(x+y)))$$

for arbitrary test function ϕ in K. If the support of ϕ is contained in the interval (a,b) we have

$$(1, \phi(x+y)) = \int_{a-y}^{b-y} \phi(x+y)\,dx$$

$$= \int_a^b \phi(x)\,dx = (1, \phi) = c,$$

where c is a constant. Thus

$$(f_n * 1, \phi) = (f_n(y), c) = 0,$$

since f_n is an odd distribution. Letting n tend to infinity we see that $f \circledast 1 = 0$. This completes the proof of the theorem.

This theorem also holds for the convolution given by definition 4, see theorem 4 of [4].

A particular case of this theorem is

$$\operatorname{sgn} x \circledast 1 = 0 \tag{4}$$

and since

$$(\operatorname{sgn} x)' \circledast 1 = 2\delta * 1 = 2$$

$$\neq 0 = (\operatorname{sgn} x \circledast 1)',$$

it follows that the equation

$$(f \circledast g)' = f' \circledast g$$

does not hold in general.

Further particular cases of the theorem are

$$x^{2r+1} \circledast 1 = 0 \;.$$

for $r = 0, \pm 1, \pm 2, \ldots$ and more generally

$$(\operatorname{sgn} x . |x|^{\lambda}) \circledast 1 = 0$$

for all λ.

Theorem 6. The convolution $x_+^\lambda \circledast x^s = 0$ for $\lambda > -1$ and $s = 0,1,2, \ldots$.

Proof. The convolution $(x_+^\lambda)_n \circledast x^s$, where

$$(x_+^\lambda)_n = x_+^\lambda \tau_n(x),$$

again exists by definition 2 and so

$$((x_+^\lambda)_n * x^s, \phi(x)) = ((y_+^\lambda)_n, (x^s, \phi(x+y))$$

for arbitrary test function ϕ in K. If the support of ϕ is contained in the interval (a,b) we have

$$(x^s, \phi(x+y)) = \int_{a-y}^{b-y} x^s \phi(x+y)\,dx$$

$$= \int_a^b (t-y)^s \phi(t)\,dt = \sum_{i=0}^{s} a_i y^i,$$

where

$$a_i = \binom{s}{i} (-1)^i (t^{s-i}, \phi(t))$$

for $i = 0,1, \ldots, s$. Thus

$$((x_+^\lambda)_n * x^s, \phi(x)) = \sum_{i=0}^{s} a_i((y_+^\lambda)_n, y^i)$$

$$= \sum_{i=0}^{s} a_i\left[\int_0^n y^{\lambda+i}\,dy + \int_n^{n+n^{-n}} y^{\lambda+i}\,dy\right]$$

$$= \sum_{i=0}^{s} \frac{a_i n^{\lambda+i+1}}{\lambda+i+1} + O(n^{-n+s+\lambda})$$

and it follows that

$$\underset{n\to\infty}{\text{N-lim}}\, ((x_+^\lambda)_n * x^s, \phi(x)) = 0.$$

The result of the theorem follows. This completes the proof of the theorem.

Corollary 1. The convolutions $x_-^\lambda \circledast x^s$, $|x|^\lambda \circledast x^s$ and $(\operatorname{sgn} x . |x|^\lambda) \circledast x^s$ exist and

$$x_-^\lambda \circledast x^s = |x|^\lambda \circledast x^s = (\operatorname{sgn} x . |x|^\lambda) \circledast x^s = 0$$

for $\lambda > -1$ and $s = 0,1,2, \ldots$.

Proof. The first result follows on replacing x by $-x$ in the theorem. The other two results follow on noting that

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad \operatorname{sgn} x . |x|^\lambda = x_+^\lambda - x_-^\lambda,$$

the convolution product being distributive with respect to addition.

Corollary 2. The convolutions $x_-^\lambda \circledast x_+^s$, $|x|^\lambda \circledast x_+^s$ and $(\operatorname{sgn} x . |x|^\lambda) \circledast x_+^s$ exist and

$$x_-^\lambda \circledast x_+^s = (-1)^{s+1} B(\lambda+1, s+1) x_-^{\lambda+s+1}, \tag{5}$$

$$|x|^\lambda \circledast x_+^s = \begin{cases} B(\lambda+1,s+1) \operatorname{sgn} x . |x|^{\lambda+s+1}, & \text{even } s, \\ B(\lambda+1,s+1) |x|^{\lambda+s+1}, & \text{odd } s, \end{cases} \tag{6}$$

$$(\operatorname{sgn} x . |x|^\lambda) \circledast x_+^s = \begin{cases} B(\lambda+1,s+1)|x|^{\lambda+s+1}, & \text{even } s, \\ B(\lambda+1,s+1) \operatorname{sgn} x . |x|^{\lambda+s+1}, & \text{odd } s \end{cases} \tag{7}$$

for $\lambda > -1$ and $s = 0,1,2, \ldots$.

Proof. The convolution $x_+^\lambda * x_+^s$ exists by definition 2 and

$$x_+^\lambda * x_+^s = \int_{-\infty}^{\infty} (x-t)_+^\lambda t_+^s \, dt = B(\lambda+1,s+1) x_+^{\lambda+s+1}, \tag{8}$$

where B denotes the beta function. Replacing x by $-x$ we have

$$
\begin{aligned}
x_-^{\lambda} * x_-^{s} &= B(\lambda+1,s+1)\,x_-^{\lambda+s+1} \\
&= (-1)^s x_-^{\lambda} * (x^s - x_+^{s}) \\
&= (-1)^s x_-^{\lambda} \circledast x^s - (-1)^s x_-^{\lambda} \circledast x_+^{s}
\end{aligned}
$$

and equation (5) follows on using equations (5) and (8).

Further

$$
\begin{aligned}
|x|^{\lambda} \circledast x_+^{s} &= (x_+^{\lambda} + x_-^{\lambda}) \circledast x_+^{s} \\
&= x_+^{\lambda} * x_+^{s} + x_-^{\lambda} \circledast x_+^{s}
\end{aligned}
$$

and equation (6) follows on using equations (5) and (8).

Finally

$$
\begin{aligned}
(\operatorname{sgn} x . |x|^{\lambda}) \circledast x_+^{s} &= (x_+^{\lambda} - x_-^{\lambda}) \circledast x_+^{s} \\
&= x_+^{\lambda} * x_+^{s} - x_-^{\lambda} \circledast x_+^{s}
\end{aligned}
$$

and equation (7) follows on using equations (5) and (8).

Corollary 3. The convolutions $x_+^{\lambda} \circledast |x|^{2s-1}$, $x_-^{\lambda} \circledast |x|^{2s-1}$, $|x|^{\lambda} \circledast |x|^{2s-1}$ and $(\operatorname{sgn} x . |x|^{\lambda}) \circledast |x|^{2s-1}$ exist and

$$x_+^{\lambda} \circledast |x|^{2s-1} = 2B(\lambda+1,2s)\,x_+^{\lambda+2s}, \tag{9}$$

$$x_-^{\lambda} \circledast |x|^{2s-1} = 2B(\lambda+1,2s)\,x_-^{\lambda+2s}, \tag{10}$$

$$|x|^{\lambda} \circledast |x|^{2s-1} = 2B(\lambda+1,2s)\,|x|^{\lambda+2s}, \tag{11}$$

$$(\operatorname{sgn} x . |x|^{\lambda}) \circledast |x|^{2s-1} = 2B(\lambda+1,2s)\operatorname{sgn} x . |x|^{\lambda+2s} \tag{12}$$

for $\lambda > -1$ and $s = 1,2, \ldots$.

P r o o f. $x_+^\lambda \circledast |x|^{2s-1} = x_+^\lambda \circledast (2x_+^{2s-1} - x^{2s-1})$

$$= 2x_+^\lambda * x_+^{2s-1} - x_+^\lambda \circledast x^{2s-1}$$

and equation (9) follows on using equation (8) and theorem 6. Equation (10) follows on replacing x by $-x$ in equation (9) and equations (11) and (12) now follow immediately from equations (9) and (10).

C o r o l l a r y 4. The convolutions $x_+^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s})$, $x_-^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s})$, $|x|^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s})$ and $(\operatorname{sgn} x \,.\, |x|^\lambda) \circledast (\operatorname{sgn} x \,.\, x^{2s})$ exist and

$$x_+^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s}) = 2\,B(\lambda+1, 2s+1)x_+^{\lambda+2s+1},$$

$$x_-^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s}) = -2\,B(\lambda+1, 2s+1)x_-^{\lambda+2s+1},$$

$$|x|^\lambda \circledast (\operatorname{sgn} x \,.\, x^{2s}) = 2\,B(\lambda+1, 2s+1)\operatorname{sgn} x \,.\, |x|^{\lambda+2s+1}, \tag{13}$$

$$(\operatorname{sgn} x \,.\, |x|^\lambda) \circledast (\operatorname{sgn} x \,.\, x^{2s}) = 2\,B(\lambda+1, 2s+1)|x|^{\lambda+2s+1}$$

for $\lambda > -1$ and $s = 0, 1, 2, \ldots$.

P r o o f. The results follow as above on noting that

$$\operatorname{sgn} x \,.\, |x|^{2s} = 2x_+^{2s} - x^{2s}.$$

A particular case of equation (13) is

$$x^{2r} \circledast \operatorname{sgn} x = \frac{2x^{2r+1}}{2r+1}$$

for $r = 0, 1, 2, \ldots$. This result also holds for the convolution given by definition 4, see theorem 5 of [2].

The case $r = 0$ is

$$1 \circledast \operatorname{sgn} x = 2x$$

and comparing this equation with equation (4) we see that the convolution product is not in general commutative, even if the convolution of two distributions exists in either order.

Theorem 7. The convolutions $x^{-r} \circledast x^{s}$ and $x^{s} \circledast x^{-r}$ exist and

$$x^{-r} \circledast x^{s} = x^{s} \circledast x^{-r} = 0$$

for $s = 0,1, \dots ,r-1$ and $r = 1,2, \dots$.

This theorem was proved in [2] for the convolution given by definition 4. The proof can easily be adapted for the convolution given by definition 6 and so we omit the proof.

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REZIME

NEUTRIKSI I KONVOLUCIJA DISTRIBUCIJA

Data je jedna nova definicija konvolucije distribucija i ispitane su neke njene osobine.

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