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A CONSTRUCTION OF ALL THE NON-ISOMORPHIC
NON-SIMPLE MATROIDS ON 9 ELEMENTS

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ABSTRACT

All the 6705 non-isomorphic non-simple matroids on 9 elements are constructed (without computer assistance). The construction is based on the catalogue [4] of all the non-isomorphic matroids on at most 8 elements. The paper is a sequel to [5]; similar methods are used, but the denotations are somewhat shortened. The paper includes a catalogue of all the 4981 loopless matroids in the considered class.

1. PRELIMINARIES

Sets are mostly denoted without brackets and commas. Throughout the paper, let "k" and "r" denote the number of atoms and the rank (of matroids) respectively.

We shall give only a few less standard definitions; the others can be found in [7].

A semisimple matroid is a loopless non-simple matroid.

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An essential flat of a matroid is a flat F which satisfies the following three additional conditions:

- a) F is cyclic (i.e. a union of circuits).
- b) F is different from the ground-set.
- c) The existence of F cannot be "predicted" by using (in other words, F is not a "minimal consequence" of) the family of all flats of lower ranks.

The essential flats, together with their ranks, are sufficient to describe a matroid and seem to make (at random) a very concise matroid representation.

All the non-isomorphic matroids with loops on 9 elements can be very easily obtained by introducing one new loop in each of the 1724 non-isomorphic general matroids on 8 elements. The latter matroids can be found in the catalogue [4], which is an extension of the catalogues contained in papers [6] and [1]. In order to finish our construction, it therefore suffices to construct all the non-isomorphic semi-simple matroids on 9 elements.

2. DENOTATIONS AND ABBREVIATIONS IN THE CATALOGUE OF SEMISIMPLE MATROIDS

In general, we shall abbreviate the matroid denotations introduced in [4]. These denotations have the form " $n=S:T$ ", where

- " S " denotes a simple matroid on k elements (=atoms), $8 \geq k \geq 4$,
- " T " denotes a collection of all the non-isomorphic semisimple matroids M_i on 9 elements which have the property that the geometric lattices of M and M_i are isomorphic,
- " n " denotes the ordinal number of the simple matroid M within the class of non-isomorphic simple matroids with the fixed pair (k,r) , where $r = \text{rank}(M)$.

2a. Denotations of simple matroids

The ground-set of simple matroids on k elements is always denoted by the first k letters of the word "abcdefgh".

The letter "S" replaces the family of essential flats of the simple matroid M , accompanied by their ranks. If the rank (F) of an essential flat F is either of the values $\text{rank}(M)-1$ (when $|F| \geq \text{rank}(M)$) or $|F| - 1$ (when $|F| < \text{rank}(M)$), then we omit the denotation of $\text{rank}(F)$, otherwise the denotation " F " is replaced by " $F(\text{rank}(F))$ ". Equivalently, $\text{rank}(F)$ is given the brackets after an essential flat F iff

$$\text{rank}(F) \leq \min\{\text{rank}(M), |F|\} - 2.$$

In order to shorten the lists of essential flats, we shall divide the denotation of a simple matroid into "blocks". In a great majority of cases, a block is one of the following:

- (i) an essential flat
- (ii) an ordinal number of a simple matroid
- (iii) a letter from the set {A,B,C,D,E}.

Blocks are separated by commas.

Block " n " is a replacement for the whole list of blocks corresponding to the simple matroid (with the same (k,r)), numerated by n .

Blocks "A", "B", "C", "D", "E" replace the first one, two, three, four, five blocks respectively, from the list of blocks corresponding to the preceding simple matroid in the catalogue.

More rarely, we shall use blocks like " nB ", " nC ", ..., which refer just to the first two, three, ..., blocks from the list of blocks denoted by " n ".

EXAMPLE. Let $(k,r)=(8,4)$ and $n=322$. Using the "chain" $322=B,efgh; 321=319B,cdfg; 319=307,bcfh,cdgh; 307=A,abfg; 306=A,afgh; 305=abcde$ - we obtain that

$$322=abcde,abfg,bcfh,cdfg,efgh.$$

Since the ranks are not explicitly given in the brackets, we can easily derive that all the five constructed essential flats have rank 3.

Very rarely, we shall also allow the blocks of the form " $n-E_1+E_2$ " or " $n-E_1-E_2+E_3+E_4$ ". They refer to the list of blocks denoted by " n ". The essential flats preceded by " $-$ " are deleted from that list and replaced by the essential flats preceded by " $+$ ".

Simple uniform matroids (with no essential flats; there exists only one uniform matroid for each (k,r)) are denoted simply by " U ".

2b. Denotations of semisimple matroids

Given a simple matroid M on k atoms, the non-isomorphic semisimple matroids in the corresponding collection " T " are denoted by some unordered words (=combinations with repetitions) of length $9-k$, over the set of atoms of M . These words are separated by commas.

A j -fold appearance of the letter " x " in the word adjoined to a semisimple matroid M_1 means that the atom x in M is replaced in M_1 by a "bundle" of $j+1$ mutually parallel elements (the same replacement is also performed within all the flats containing x ; recall that all the flats of a matroid are unions of atoms!). This looks like an "addition" of j "new" parallel elements to the atom x . There must be no intersections between the new parallel elements added to different atoms of M .

EXAMPLE. Let $(k,r)=(6,3)$ and $M=abc, ade, cef$. Denote the three new parallel elements by " g ", " h ", " i ", respectively. Then the essential flats of the corresponding semisimple matroid M_1 , denoted by the word "bba", are:

$ai, bgh(1), abcghi, adei, cef$.

Let " $y(M)$ " denote the number of non-isomorphic semisimple matroids on 9 elements corresponding to a simple matroid M on 8 elements. If $y(M) \in \{6,7\}$, then we can shorten

the denotation "T" as in the following examples:

"(bc)" replaces "a,b,d,e,f,g,h"

while

"(de)(gh)" replaces "a,b,c,d,f,g";

(briefly, each of the brackets and each of the missing elements corresponds to a special semisimple matroid).

If $y(M)=8$, then the denotation "T" is simply replaced by "8".

The denotation "n.r" in the place of "T" with a simple matroid on $k \leq 7$ atoms means that the denotation "T", which was met with the n-th rank r simple matroid on k elements, should be repeated.

It is easy to see that there is a bijection between non-isomorphic semisimple matroids corresponding to a uniform matroid on k atoms and different partitions of the natural number $9-k$ into at most k natural summands. Since the only simple matroids on k atoms, which have their ranks r in the set {2,k} - are uniform, we shall omit the corresponding semisimple matroids from the catalogue and list the semisimple matroids just for the pairs (k,r) satisfying $8 \geq k \geq 4$, $3 \leq r \leq k-1$.

Finally, there exists only one semisimple matroid of rank 1 on 9 elements.

* * *

The simple and their corresponding semisimple matroids are listed in the catalogue separately for each pair (k,r) . The ordinal numbers of simple matroids are directly copied from [6]. The denotations of semisimple matroids are lexicographically ordered for $k=8$, while the primary criterion in ordering semisimple matroids for smaller values of k was the partition of the numbers of different letters in the adjoined words (e.g. the partition corresponding to the word "dddab" is 3+1+1).

3. A CATALOGUE OF ALL THE NON-ISOMORPHIC
SEMISIMPLE MATROIDS ON 9 ELEMENTS

(k,r) = (8,3)

1=U:a 2=abc:a,d 3=abcd:a,e 4=abcde:a,f 5=2,ade:a,b,f
 6=2,def:a,g 7=5,bdf:a,c,g 8=5,afg:a,b,h 9=5,dfg:a,b,e,h
 10=5,fgf:a,b,f 11=3,aef:a,b,e,g 12=3,efg:a,e,h 13=11,agh:
 :a,b,e 14=A,deg:a,b,e,f,h 15=A,egh:a,b,e,f,g 16=A,bgh:a,c,e
 17=3,aefg:a,b,h 18=3,efgh:a 19=5,bfg,dfh:a,c 20=2,cde,aef,
 agh:a,b,c,d,g 21=B,afg,bfh:a,c,d,f,g 22=5,bef,cdf:a,g 23=5,
 afg,ceg:a,b,c,h 24=6,adg,cfg:a,b,g,h 25=11,beg,ceh:a,d,e,f
 26=B,cfh:a,b,d,e,g 27=3,aeg,afh,bef:a,b,c,e,g 28=3,ah,dgh,
 efg:a,b,e,f,h 29=11,beg,cfg:a,d,e,h 30=4,afg:a,b,f,h 31=4,
 fgh:a,f 32=2,cde,efg,afh,bgh:a,c,d,h 33=B,aef,bdf,agh:a,b,
 d,g 34=A,efg,cgh,ah:a,b,c,d,h 35=C,bfh:a,b,c,f,g 36=7,dgh,
 efg:(ab)(ef) 37=2,cde,aef,beg,cfg:a,c,d,h 38=C,adg,bfg:a,b,h
 39=3,defg,agh:a,b,d,h 40=2,aef,cde,cfg,beg,adg:a,b,h 41=5,
 bdf,cef,cdg,fgf:(be)(cd) 42=2,cde,efg,ah,bfh,dgh:a,b,d,e,g
 43=35,ah:a,b,c,d,h 44=8,bfh,ceh,dgh:a,b 45=A,bdf,beh,cef:
 :a,c,g 46=30,efh:a,b,f,g 47=3,agh,bef,ceg,dfg:(cd)(ef) 48=
 =3,ah,afg,def,dgh:a,b,e 49=11,agh,beg,cfh:a,b,d,e 50=25,
 fgh:a,d,e,f 51=26,dgh:a,e 52=11,agh,bfh,cfg:(bc)(gh) 53=4,
 afgh:a,b,f 54=abcdef:a,g 55=17,cfh,dgh:a,b,c,h 56=54,agh:
 :a,b,g 57=abcdefg:a,h 58=30,cfh,ehg:a,b,f 59=17,beh,cfh,
 dgh:a,b,h 60=11,agh,beg,bfh,cfg:a,c,d,e,f 61=3,agh,bef,ceg,
 dfg,deh:a,c,d,e,f 62=8,bdg,ceg,cdf,bef:a,h 63=D,beh:a,b,f,h
 64=40C,adh,bgh,cfh,dfg:a,b,e,h 65=62B,beh,cef,cgh:a,d,e 66=
 =3,ah,afg,bfh,cef,cgh,deg:a,b,e 67=3,aef,agh,beh,bfg,ceg,
 cfh:a,d,e 68=2,adf,agh,bde,bfh,cdg,ceh,efg:a

(k,r) = (8,4)

1=3,cdg:a,h 2=abcdef(2):a,g 3=7,cef:a,c,h 4=13,cfg:a,d,e,h
 5=16,afg:a,b,f,h 6=A,fgf:a,f 7=23,beg:a,c,d,h 8=A,ceg:a,b,h
 9=25,agh:a,b,f,g 10=70,efgh(2):a 11=A,aefg(2):a,b,h 12=31,
 egh:a,b,e,f,g 13=A,beg:a,c,e,f,h 14=A,agh:a,b,e 15=A,bgh:
 :a,c,e 16=abcde(2):a,f 17=A,afgh:a,b,f 18=A,296:a,f,h

19=abc,cde,efg,agh:a,b 20=D,bdfh:a,b 21=61,fg:h:a,b,c,f,g
 22=46,bdh:a,b,c,f,h 23=A,bdf:a,b,c,h 24=61,ceg:a,b,f,h 25=
 =A,cef:a,g 26=B,afgh:a,b,g 27=B,begh:a,c,g 28=B,cdfh:a,g
 29=25,abcgh:a,d,g 30=A,296:a,g,h 31=70,aef:a,b,e,g 32=B,
 begh:(cd)(gh) 33=B,cfgh:a,b,d,e,g 34=31,abcdgh:a,b,e,g
 35=A,aefgh:a,b,e,g 36=A,296:a,b,e,g,h 37=70,efg:a,e,h 38=
 =B,296:a,e,h 39=A,abceh:a,e,f,h 40=A,aefgh:a,b,e,h 41=
 =108,fg:h:a,b,f 42=B,bdfgh:a,b,c,f 43=B,cefgh:a,b,f 44=41,
 297:a,b,f,g 45=A,abcfgh:a,b,d,f 46=108,afg:a,b,h 47=B,bdfh:
 :a,b,c,h 48=B,begh:(de)(fg) 49=A,cegh:a,b,h 50=48,cdgh:a,
 b,c,h 51=B,cefh:a,b,h 52=46,abceh:a,b,f,h 53=A,296:a,b,h
 54=abc,adg,def:a,b,g,h 55=C,begh:a,b,c,g,h 56=B,cfgh:a,b,
 g,h 57=54,abceh:(bc) 58=A,bfgh:8 59=A,296:a,b,g,h 60=A,
 abcdgh:(bc)(ef) 61=108,bdf:a,c,g 62=B,afgh:a,b,c,f,g 63=A,
 cegh:a,b,c,f,g 64=62,begh:a,c,d,e,g 65=A,cegh:a,b,c,f,g
 66=64,cdgh:a,c,g 67=61,abcgh:a,c,d,e,g 68=B,efgh:a,c,d,e,g
 69=61,296:a,c,g,h 70=abcd(2):a,e 71=A,aefgh:a,b,e 72=A,
 aefg:a,b,e,h 73=A,efgh:a,e 74=72,befh:a,c,e,g 75=B,cegh:a,
 d,e,f 76=B,dfgh:a,e 77=70,297:a,e,g 78=B,aegh:a,b,e,f,g
 79=A,efgh:a,e,g 80=78,befh:(ab)(cd) 81=77,abcdgh:a,e 82=
 =B,efgh:a,e 83=70,296:a,e,h 84=abc,def:a,g 85=B,adgh:a,b,g
 86=B,begh:a,c,g 87=B,cfgh:a,g 88=84,abcgh:a,d,g 89=B,defgh:
 :a,g 90=A,adefg:a,b,d,g,h 91=95,defgh:a,d,f,g 92=84,abcdg:
 :a,d,e,g,h 93=B,aegh:(bc) 94=B,bfgh:(ab)(ef) 95=92,abceh:
 :a,d,f,g 96=B,afgh:a,b,d,f,g 97=92,adefh:a,b,g 98=B,begh:
 :a,b,c,g 99=B,cfgh:a,b,g 100=84,297:a,g 101=B,adgh:a,b,g
 102=B,begh:a,c,g 103=B,cfgh:a,g 104=100,abcgh:a,d,g 105=B,
 defgh:a,g 106=84,abcdgh:a,d,e,g 107=A,296:a,g,h 108=abc,
 ade:a,b,f 109=B,afgh:a,b,f 110=A,bfgh:a,b,c,d,f 111=A,bdfg:
 :a,b,c,f,h 112=B,befh:(de)(gh) 113=A,cefh:a,b,f,g 114=112,
 cdgh:a,b,c,f,h 115=A,cdfh,cefg:a,b,f,g 116=114,cefg:a,b,f,h
 117=111,cefg:a,b,f,h 118=109,bdfg:a,b,c,f,h 119=111,cfgh:
 :(fg) 120=118,befh:(de)(gh) 121=112,cfgh:(de)(gh) 122=113,
 afgh:a,b,f,g 123=112,cdfh,afgh:(bd)(ce) 124=117,begh,afgh,
 cdfh:a,b,f,h 125=C,cdgh:a,b,f,g 126=117,afgh:a,b,f,h 127=
 =A,befh:(be)(cd) 128=111,cegh,befh,afgh:a,b,c,f,g 129=108,
 abcgh:a,b,d,f,h 130=B,bdfh:8 131=A,dfgh:(bc)(fg) 132=130,
 cegh:a,b,d,f,h 133=A,cefh:(bc)(de) 134=129,befh,bdgh:(de)(fg)

135=130,cdgh:(bc)(fg) 136=131,begh:8 137 = B,cefh:(bc)(fg) 138=
 =135,begh:8 139=B,cefh:a,b,d,f,h 140=129,adefh:a,b,f,g
 141=B,bdgh:a,b,c,f,g 142=B,cegh:a,b,f,g 143=108,bdfgh:a,b,
 c,f 144=B,cefg:a,b,c,f,h 145=108,297:a,b,f,g 146=B,afgh:a,
 b,f,g 147=A,bfgh:(de)(gh) 148=A,bdgh:a,b,c,f,g 149=B,afgh:
 :a,b,c,f,g 150=147,cdgh:(gh) 151=148,cegh:a,b,f,g 152=B,
 afgh:a,b,f,g 153=145,abcgh:a,b,d,f,g 154=B, degh:a,
 b,d,f,g 155=108,abcfgh:a,b,d,f 156=A,296:a,b,f,h 157=abc:
 :a,d 158=A,efgh:a,d,e 159=A,adef:a,b,d,g 160=B,efgh:a,b,d,
 e,g 161=A,adgh:a,b,d,e 162=A,bdeg:a,c,d,f,h 163=A,bdgh:a,
 c,d,e 164=efgh,161:a,b,d,e 165=A,162:(ab)(fg) 166=A,163:a,
 c,d,e 167=161,bdeg:(eg)(fh) 168=A,befg:(ef) 169=162,cdfg:
 :a,d,e,h 170=A,cdeh:a,d,f 171=A,cdfh:a,b,d,e,g 172=A,cfgh:
 :a,c,d,f,h 173=efgh,167:(eg)(fh) 174=A,169:a,d,e,h 175=A,
 170:a,d,e,f 176=A,171:a,b,d,e,g 177=167,bdfh:a,c,d,e
 178=168,bdeh:a,c,d,f,g 179=167,cdfh:a,b,d,e 180=
 =A,cdfg:(bc)(ef) 181=168,cegh:a,b,d,e,f 182=167,cefg:8
 183=168,cdeh:8 184=A,cefh:a,b,d,e,g 185=efgh,177:a,c,d,e
 186=A,179:a,b,d,e 187=A,180:(bc)(ef) 188=177,cdfg:a,c,d,e,f
 189=184,bdeh:8 190=178,cdeg:a,c,d,f,g 191=161,bdeh,bfgh,
 cefg:a,c,d,e,f 192=168,bdfh,cdeg:8 193=177,cefg:(ab)(fg)
 194=188,efgh:a,c,d,e,f 195=A,cdeh:a,d,e, 196=A,cefh:(ab)
 (eh) 197=163,aegh,cdfg,befg,cdeh:a,d,f 198=159,aegh,bdfg,
 bdeh,cdeg,cfgh:a,c,d,f,g 199=177,cdeh,cdfg,efgh:a,d,e 200=
 =abc,abcde:a,d,f 201=B,afgh:a,b,d,f 202=A,dfgh:a,d,e,f
 203=219,aefh:a,b,d,f,g 204=A,bdfh:(ab)(gh) 205=A,befh:a,c,d,
 f,g 206=A,befg:a,c,d,f,h 207=A,bfgh:(fg) 208=A,efgh:(bc)
 (fg) 209=203,bdfh:8 210=A,bdgh:8 211=A,bfgh:(de)(gh) 212=
 =204,cdgh:a,d,e,f 213=A,cegh:(ab)(gh) 214=A,cefg:8 215=A,
 cfgh:(ab)(gh) 216=cfgh,206:a,c,d,f,h 217=A,205:a,c,d,f,g
 218=204,efgh:(ab)(gh) 219=200,adfg:(bc)(fg) 220=befg,209:a,
 c,d,f,g 221=A,210:a,c,d,f,h 222=209,cefg:a,b,d,f,g 223=A,
 cegh:8 224=203,begh,cdgh:a,b,d,f,g 225=cfgh,209:8 226=A,
 210:8 227=209,cdgh:8 228=212,efgh:a,d,e,f 229=cfgh,220:a,
 c,d,f,g 230=A,221:a,c,d,f,h 231=220,cdgh:(ab)(gh) 232=205,
 aegh,bdgh,cdfh:8 233=227,begh,cefg:a,d,f 234=200,defg:a,d,
 f,h 235=B,afgh:a,b,d,f,h 236=A,adf:(bc) 237=B,bdgh:(ab)
 (fg) 238=A,befh:(ab)(de) 239=A,begh:a,c,d,f,h 240=A,aegh:
 :a.b.d.f.h 241=A,bfgh:8 242=240.befh:8 243=A,bfgh:(de)(fg)

244=238,cfgh:(ab)(de) 245=A,cdgh:8 246=cfg, 239:a,c,d,f,h
 247=A,237:(ab)(fg) 248=242,bdgh:a,c,d,f,h 249=240,bdgh,cefh:
 :a,b,d,f,h 250=B,cfgh:8 251=B,befh:a,c,d,f,h 252=abc,adefg:
 :a,b,d,h 253=B,bdeh:(de)(fg) 254=B,bfgh:a,b,c,d,h 255=A,
 cfg: a,b,d,h 256=A,cdfh:(bc)(ef) 257=B,bfgh:(df)(eg) 258=
 =B,cegh:a,b,d,h 259=abc,defgh:a,d 260=200,abcfg:a,d,h 261=
 =B,adf:h:a,b,d,e,h 262=B,aegh:a,b,d,h 263=A,bdgh:(ab)(fg)
 264=A,begh:a,c,d,h 265=262,bdgh:(dg)(ef) 266=263,cefh:a,b,d,
 e,h 267=265,hefh:a,c,d,h 268=A,cefh:a,b,d,h 269=260,defg:
 a,d,h 270=B,adf:h:a,b,d,e,h 271=B,aegh:a,b,d,h 272=A,bdgh:
 (ab)(fg) 273=A,begh:a,c,d,h 274=271,bdgh:(dg)(ef) 275=272,
 cefh:a,b,d,e,h 276=274,befh:a,c,d,h 277=A,cefh:a,b,d,h
 278=260,defh:a,d,f,g,h 279=B,adgh:(bc) 280=B,begh:(ab)(de)
 281=200,adfg:h:a,b,d,e,f 282=B,befg:(fg) 283=B,cefh:(bc)(gh)
 284=200,defgh:a,d,f 285=B,abcfg:a,d,h 286=abc,abcdef:a,d,g
 287=B,adgh:a,b,d,e,g 288=B,begh:a,c,d,f,g 289=A,efgh:a,b,d,
 e,g 290=288,cfgh:a,d,g 291=286,degh:a,d,f,g 292=A,abcgh:a,
 d,g 293=B,degh:a,d,f,g 294=abc,adefgh:a,b,d 295=A,296:a,d,h
 296=abcdefg:a,h 297=abcdef:a,g 298=A,abgh:a,c,g 299=B,cdgh:
 :a,e,g 300=B,efgh:a,g 301=305,abfg:h:a,c 302=B,cdfg:a,c,e
 303=B,cefh:a,c,d 304=B,degh:a,c 305=abcde:a,f 306=A,afgh:
 :a,b,f 307=A,abfg:a,c,f,h 308=B,cfgh:a,c,d,f,h 309=A,acf:h:
 :a,b,d,f,g 310=B,dfgh:(bc)(gh) 311=307,cdfh:a,e,f,g 312=B,
 efgh:a,e,f,g 313=307,cdfg:a,e,f,h 314=B,efgh:a,e,f,h 315=
 =309,adgh:a,b,e,f 316=B,efgh:a,b,e,f 317=309,bcgh:a,d,f
 318=B,efgh:a,d,e,f 319=307,bcfh,cdgh:a,b,e,f,g 320=C,efgh:
 :a,b,e,f,g 321=319B,cdfg:(ad)(bc) 322=B,efgh:(ad)(bc) 323=
 =309,degh:a,b,d,f,g 324=A,defg:(de) 325=321,adgh:a,e,f,h
 326=B,efgh:a,e,f,h 327=321,adf:h:a,e,f,g 328=B,efgh:a,e,f,g
 329=319B,acgh,adf:h:(bc)(fh) 330=C,efgh:(bc)(fh) 331=319,defg:
 :a,b,c,f,h 332=321,degh:8 333=A,defh:a,b,c,f,g 334=329B,
 defg:a,c,d,f,h 335=319B,bdgh,defh:8 336=327,acgh:a,b,e,f,g
 337=B,efgh:a,b,e,f,g 338=331,aefh:a,b,c,f,g 339=319B,cefg,
 cdgh,defh:8 340=333,bdgh:a,b,c,f,g 341=319,defh,begh:(ce)
 (fg) 342=333,begh:8 343=313,acf:h, bdfh,adgh,bcgh:a,e,f 344=
 =E,efgh:a,e,f 345=313,bcfh,defh,aegh,bdgh:a,b,c,f,g 346=307,
 cefg, befh,cdfh,bcgh,degh:(ce)(fg) 347=305,acfg,befg,adf:h,
 bcfh,aegh,bdgh:a,c,f 348=U:a 349=abcd:a,e 350=A,abef:a,c,g

351=A,efgh:a 352=350,abgh:a,c 353=A,aceg:a,b,d,h 354=A,
 cdef:a,g 355=A,cdgh:a,e 356=353,adfg:a,b,h 357=352,aceg:a,
 b,c,d 358=A,cdef:a,c,g 359=353,bceh:a,d 360=A,bcfg:a,d,e,h
 361=A,bdfh:a,c,g 362=355,efgh:a 363=357,acfh:a,b,d,e 364=
 =356,bceh:a,b,d,h 365=A,bcfg:a,b,d,h 366=357,bceh:a,c,d,g
 367=A,bdfh:a,c 368=A,cdef:(ce)(df) 369=358,cdgh:a,e 370=
 =361,cdef:a,g, 371=A,cdgh:a,e 372=363,adeh:a,b,d 373=A,
 bceh:a,b,d,e,f 374=A,bdeg:(eg)(fh) 375=A,efgh:a,b,d,e 376=
 =365,bdeg:a,c,h 377=364,bdfh:a,c,g 378=366,cdef:a,d 379=
 357,bcfg,cdef:(ab)(ef) 380=367,cdef:a,c,g 381=368,cdgh:a,b,f
 382=369,efgh:a 383=371,efgh:a 384=372,adfg:a,b 385=A,bceh:
 :a,b,d 386=A,bcfg:a,b,d,e,f 387=373,bcfg:a,d,e 388=A,bdeg:
 :a,c,d,g 389=A,efgh:a,b,d,e,f 390=374,bdfh:a,c,e 391=A,
 cdef:(bc)(fg) 392=A,efgh:(eg)(fh) 393=376,cdef:a,h 394=377,
 cdgh:a,e 395=380,cdgh:a,e 396=381,efgh:a,b 397=366,cdgh,
 efgh:a,c,d 398=adfg,bceh,bcfg,bdeg,bdfh,cdef,cdgh,efgh:a,b,d
 399=398-bceh+adeh:a,b,d,e,f 400=398-bdeg+adeh:(eg)(fh) 401=
 =398-efgh+adeh:a,b,d,e 402=400-adfg-bcfg+abgh+acfh:a,c,h
 403=402-bceh-bdfh+bcfg+bdeg:a,c,g 404=405-bceh+bcfg:a,d
 405=400-bcfg-cdef+acfh+bdeg:(ab)(ef) 406=405-bdfh+bcfg:a,c,g
 407=405-cdgh+bcfg:a,b,f 408=407-efgh+aceg:a 409=408-aceg-
 -bdfh+abgh+cdef:a 410=401,efgh:a,b,d,e 411=403,bdfh:a,b,d,h
 412=402,bdeg:a,b,d,h 413=404,cdef:a,c,d,g 414=406,cdef:a,c
 415=407,cdgh:(ce)(df) 416=408,efgh:a,e 417=406,abgh:a,g
 418=409,efgh:a,e 419=411,bceh:a,b,h 420=413,bceh:a,b,c,d
 421=416,cdgh:a,c,g 422=411,adfg:a,d 423=412,adfg:a,d,e,h
 424=417,cdef:a,c,g 425=409,aceg,bdfh:a 426=421,cdef:a,c
 427=419,adfg:a,b,d,h 428=421,abgh:a,g 429=425,efgh:a,e 430=B,
 cdgh:a,c,g 431=425,abef,cdgh:a 432=C,efgh:a,e 433=B,abcd:a
 434=349,aefg:a,b,h 435=B,befh:a,c,e,g 436=434,abeh,bcfg:a,
 c,d,f 437=435,cdgh:a,c 438=B,bdeg,aceh:a,c,e,f 439=C,bcfg:
 :a,b,c,d 440=B,aceh,adfh:a 441=435,cegh:a,d,e,f 442=B,dfgh:
 :a,e 443=A,adfh:8 444=B,bdfg:a,c,d,f 445=A,bdgh:8 446=A,
 bcfg:8 447=A,bdeg:8 448=445,cdfg:a,d,e,f 449=447,bcfg:a,b,
 c,e 450=445,bcfg:8 451=A,cdef:8 452=450,cdef:8 453=435,
 acgh:(ef) 454=B,bdgh:a,c,e,g 455=A,cdef:a,b,d,e 456=A,bdeg:8
 457=A,adeh,bceg:8 458=455,bdgh:a,e 459=adeh,bceg,454:a,c,e,
 f,g 460=B,455:a,b,d,e,f 461=A,458:a,e,f 462=456,cdfh:a,c

463=A,adfh:8 464=B,bcfg:8 465=B,cdeh:a 466=435,aceh,bcfg:
 :a,c,d,e,g 467=B,bdeg:a,c,e,f,g 468=466,adfh:a,b,d,e 469=
 =434,abeh,bcfg,acfh,bdef:a,c,d 470=abgh,434:a,b,c,h 471=A,
 abcd,cegh:a,c,d,f 472=470,cdef:a,b,c,h 473=A,adeh:a,b,c,h
 474=471B,adeh,befh:a,b,c,d,g 475=471,adeh:a,b,c,f 476=473,
 acfh:a,b,h 477=474,acfh:a,b,c,g 478=473,bdeg:a,b,c,h 479=
 =475,bdeg:a,c,f 480=473,bcfg:a,b,c,d,h 481=475,bcfg:a,b,c,
 d,f 482=476,cdef:a,b,c,h 483=477,cdef:a,b,c,g 484=475,acfh,
 cdef:a,b,c,d,f 485=480,cdef:a,b,c,h 486=482,bcfg:a,b,c,h
 487=483,bcfg:a,b,d,g 488=486,bdeg:a,b,h 489=abgh,435:a,c,e,g
 490=A,abcd,cegh,aefg:8 491=C,dfgh:a,c,e,g 492=489,cdef:a,c,
 e,g 493=490,cdef:a,b,e,f 494=489,adeh:8 495=490,adeh:(bd)
 (eg) 496=abcd,befh,abgh,adeh,cegh:8 497=D,dfgh:(bd)(eg)
 498=494,acfh:(cd)(ef) 499=496,acfh:(bc)(fg) 500=494,bdeg:
 :(ab)(gh) 501=495,bdeg:(bd)(eg) 502=bcfg,494:a,c,e,g 503=
 A,495:8 504=A,496:8 505=A,491,adeh:a,c,e,g 506=cdef,498:
 :(cd)(ef) 507=A,495,acfh:8 508=A,499:8 509=508-befh+dfgh:
 :(cd)(ef) 510=cdef,502:a,c,e,g 511=A,503:a,b,e,f 512=acfh,
 510:8 513=A,511:a,b,e,f 514=A,497,bcfg,cdef:a,b,e,f 515=
 =506,bcfg,bdeg:a,c,e,g 516=436,acfh:8 517=B,bdeg:a,c,d,f
 518=489,cdgh:a,c 519=490,bdfh:a,b,c,d 520=491,abef:a,c
 521=cdef,518:a,c 522=A,519:a,b 523=adeh,518:a,b,c,d 524=A,
 519:a,b,c, 525=496,adfg:a,b,c,d 526=497,aceg:a,b,c 527=
 =acfh,523:a,b,c 528=A,525:a,b,d 529=bdeg,523:a,c,d 530=A,
 524:a,b,c 531=bcfg,523:a,c 532=A,524:a,b,c,d 533=A,525:a,
 b,c,d 534=A,520,adeh:a,c 535=cdef,527:a,b,c 536=A,acfh,
 524:a,b,c,d 537=B,525:a,b,c,d 538=cdef,520,acfh,adeh:a,b,c
 539=A,531:a,c 540=A,532:a,b 541=bcfg,535:a,b,c,d 542=A,536:
 :a,b 543=A,526,cdef,acfh:a,b, 544=bdeg,541:a,c 545=A,518,
 aceh:a,c,e,f 546=545-abgh+cdef:a,c,e,f 547=545,cdef:a,c,e,f
 548=A,bcfg:a,b,c,d 549=bcfg,546:a,b,c,d 550=A,547:a,b,c,d
 551=550-cdef+adfh:a,c 552=548,adfg,cdef:a 553=491,befh:(cd)
 (ef) 554=489,dfgh:(ab)(gh) 555=553,cdef:(cd)(ef) 556=A,
 adeh:(bd)(eg) 557=dfgh,495:8 558=A,494:(bd)(eg) 559=acfh,
 553,adeh:a,b,e,h 560=A,557:(cd)(ef) 561=bdeg,556:(bd)(eg)
 562=A,558:a,c,e,f 563=bcfg,556:8 564=A,558:8 565=559,cdef:
 :(cd)(ef) 566=565-dfgh+aefg:(cd)(ef) 567=cdef,558,acfh:8

568=A, 563:(bd)(eg) 569=bcfg, 565:(bd)(eg) 570=560, bcfg, cdef:
 :(ac)(fh) 571=569, bdeg:a,b,e,h 572=553, aefg:a,c,e,g 573=B,
 cdef:a,e 574=A, adeh:(bd)(eg) 575=B, acfh:a,b,e,h 576=574,
 bdeg:a,c,e,f 577=A, bcfg:a,c,e,g 578=cdef, 575:(cd)(ef) 579=
 =A, 577:a,e 580=578, bcfg:a,b,e,f 581=B, bdeg:a,e 582=441,
 adfh, abgh:8 583=B, cdef:8 584=B, abgh:8 585=bdfg, 582:a,c,d,f
 586=A, 583:a,c,d,f 587=B, abgh:a,c,d,f 588=bcfg, 582:8 589=A,
 583:8 590=B, abgh:8 591=bdeg, 582:8 592=A, 583:8 593=B, abgh:
 :8 594=588, bdeg:8 595=B, cdef:a,b,c,e 596=453, adeh:8 597=
 =A, bcfg:8 598=B, adeh:8 599=596, bdgh:8 600=B, bcfg:a,c,e,g
 601=596, cdef:8 602=B, bcfg:a,b,d,e 603=bdeg, 596:8 604=A, 597:
 :8 605=B, adeh:8 606=bdgh, 601:a,b,e,f 607=A, 602:a,e 608=
 =603, cdfh:a,b,c,e 609=B, bcfg:a,c 610=435, aceh, bcfg, abgh:a,
 c,d,e,g 611=C, cdef:a,c,d,e,g 612=B, abgh:a,c,d,e,g 613=610B,
 bdeg, abgh:a,c,e,f,g 614=B, cdef:a,c,e,f,g 615=B, abgh:a,c,e,
 f,g 616=610, adfh:8 617=B, cdef:a,b,d,e

$(k, r) = (8, 5)$

1=U:a 2=abcde:a,f 3=A, abcfg:a,d,h 4=2, abfgh:a,c 5=3, abdfh:
 :a,c,e 6=3, adefg:a,b,h 7=3, cdefh:a,c,f,g 8=3, defgh:a,d,h
 9=5, abegh:a,c 10=2, abefg, acefh, cdefg:a,b,e,h 11=4, acdfg,
 acefh:a,b,c,d 12=3, adefh, bdfgh:a,c 13=5, cdfgh:a,c,e,f,g 14=
 =5, defgh:a,c,d,e,h 15=9, adefg:a,b,c,d 16=5, acegh, adefg:a,b,h
 17=5, acdgh, acefh, adefg:a,b,c 18=5, acegh, defgh:a,b,d,h 19=9,
 defgh:a,c,d,h 20=12, adefg:a,b,c,d,e 21=14, acdgh:(bc)(fg) 22=
 =6, bdefh, cdfgh:a,b,d,e,h 23=5, aefgh, bdefg, cdfgh:(ad)(ce) 24=
 =3, abdgh, bdefh, cdfgh, aefgh:a,b,c,f,g 25=D, adefh:a,b,c,d,e
 26=3, abefh, bdfgh, adegh, cdefg:a,c 27=C, acdfh, adefg:a,b,c 28=
 =3, cdefg, acdfh, acegh, bcdgh, bcefh:a,c 29=B, abegh, acdgh, acefh,
 bcfh:a,b,c,d 30=3, abefh, acegh, adfgh, bcdgh, bdefg:a,b,c,d
 31=3, abdgh, acdfh, aefgh, bdefh, cdefg:a,b,c 32=29B, acefh, adfgh,
 bcdgh, bdefh:a 33=abcdef:a,g 34=2, abcfg:a,d,f 35=3, adefgh:
 :a,b,d,h 36=4, cdefgh:a,c 37=3, abdfh, cdefgh:a,c,e 38=C, abegh:
 :a,c 39=65, abcdgh:a,e 40=B, aefgh:a,b,e 41=82, abgh:a,c 42=
 =abcdefg:a,h 43=abcd:a,e 44=A, abefg:a,c,e,h 45=A, aefgh:a,
 b,e 46=44, acefh:a,b,d,e,g 47=A, cefgh:a,c,d,e,h 48=A, cdefg:
 :a,e,h 49=A, cdefh:a,e,g 50=46, adefg:a,b,e,f 51=44, acfgh,

defgh:a,b,d,e,f 52=48,acefh:a,b,e,g,h 53=46,bcegh:a,d,e,f
 54=49,acegh:a,b,e,f,g 55=46,bcfgh,defgh:a,d,e,f 56=A,bdegh,
 cdefg:a,e,f,h 57=53,cdefg:(ab)(fg) 58=46,bdefh,cdefg:a,e,g
 59=50,bcfgh:(bc)(gh) 60=54,bdfgh:a,e 61=43,acefh,bcfgh,
 adfgh,bdefg,cdegh:a,c,e,f,g 62=50,bdefh,cdefg:a,b,e,f,g 63=
 =C,bcfgh:a,e,g 64=A,bcegh:a,e,f 65=43,33:a,e,g 66=A,abefgh:
 :a,c,e 67=65,abegh:a,c,e,f,g 68=A,aefgh:a,b,e,g 69=44,
 cdefgh:a,c,e,h 70=65,acegh,bcfgh:a,c,d,e,g 71=67,cdegh:a,e,
 f,g 72=A,cefgh:(ab)(gh) 73=A,cdfgh:a,e,g 74=65,abfgh,acegh,
 bdegh:a,c,e,f,g 75=67,acfgh,defgh:a,b,d,e,g 76=B,bdfgh,
 cdegh:a,e,g 77=abefgh,82:a,c,e,g 78=cdefgh,A,103:a,e 79=A,
 abcd,cdeh,abefg:a,c,e,f,h 80=106,efgh:a,e,g 81=abcd,42:a,e,h
 82=43,abef:a,c,g 83=B,acegh:a,b,c,d,g 84=A,cdegh:a,c,f,g
 85=83,adfgh:a,b,c,g 86=A,bcfgh:a,c,d,e,g 87=A,bdfgh:a,c,g
 88=84,acfgh:(gh) 89=82,acfgh,adegh,bcegh:a,b,c,d,g 90=83,
 bdegh,cdfgh:a,c,e,f,g 91=85,bcfgh,bdegh:a,c,g 92=106,cdefgh:
 :a,c,g 93=82,42:a,c,g,h 94=43,aefg:a,b,h 95=B,bcefh:a,b,d,h
 96=B,bdegh:a,b,c,h 97=B,cdfgh:a,b,h 98=94,abcdeh:a,b,e,f,h
 99=B,bcfgh:(bc)(fg) 100=94,abeh:a,b,c,h 101=B,cdfgh:a,b,c,h
 102=94,42:a,b,h 103=43,efgh:a 104=B,33:a,e,g 105=82,efgh:
 :a,c 106=A,cdef:a,g 107=B,acegh:a,b,g 108=B,adfgh:a,b,c,g
 109=A,bdfgh:a,g 110=108,bcfgh:a,b,g 111=B,bdegh:a,g 112=
 =106,42:a,g,h 113=82,cegh:a,c,d,g 114=B,adfgh:a,b,c,d,g
 115=106,cegh:a,c,d,g 116=82,aceg:a,b,d,h 117=A,cdeg:a,e,f,h
 118=115,dfgh:a,c 119=116,adfg:a,b,h 120=A,bcfg:a,d,e,h
 121=82,bdfg,cefg:a,b,c,h 122=106,bceg,bdfg:a,b,c,h 123=120,
 defg:a,d,h 124=80,abgh,cdgh:a 125=122,adeg:a,b,h 126=119,
 bcfg,bdeg,cdef:a,h 127=abcde(3):a,f 128=A,abfgh:a,c,f 129=
 =B,cdfgh:a,e,f 130=127,42:a,f,h 131=43,aefgh(3):a,b,e 132=
 =A,abefg(3):a,c,e,h 133=82,cdefg(3):a,c,g,h 134=abcdef(3):
 :a,g 135=abc:a,d 136=A,adefg:a,b,d,h 137=A,defgh:a,d 138=
 =136,befgh:a,c,d,e 139=135,33:a,d,g 140=136,abcdeh:a,b,d,f,h
 141=139,defgh:a,d,g 142=136,abcdgh,bdefh:a,c,d,e,g 143=138,
 cdegh:a,d,f 144=139,adegh,bdfgh,cefh:a,d,g 145=139,abcdgh:
 :a,d,e 146=B,aefgh:a,b,d,e 147=A,defgh:a,d,e 148=abc,42:a,
 d,h 149=A,defg:a,d,h 150=A,adef:a,b,d,g 151=149,adefgh:a,b,
 d,h 152=A,abcdeh:a,d,f,h 153=150,bdegh:(de)(gh) 154=A,bdfgh,
 cdegh:a,b,d,e,g 155=A,abcdgh:a,b,d,e,g 156=145,efgh:a,d,e

157=150,adefgh:a,b,d,g 158=155,befgh:(ef)(gh) 159=42,150:a,
 b,d,g,h 160=A,149:a,d,h 161=150,bdeg:a,c,d,f,h 162=A,bdgh:
 :a,c,d,e 163=B,cefgh:a,c,d,e 164=150,adgh:a,b,d,e 165=149,
 adeh:a,b,d,f,h 166=164,efgh:a,b,d,e 167=161,cdfg:a,d,e,h
 168=135,127:a,d,f 169=A,defgh(3):a,d 170=168,defgh:a,d,f
 171=A,adfgh:a,b,d,e,f 172=B,befgh:a,c,d,f 173=168,abcfgh:a,
 d,f 174=B,defgh:a,d,f 175=168,42:a,d,f,h 176=135,adefg(3):
 :a,b,d,h 177=168,dfgh:a,d,e,f 178=150,abcgh(3):a,b,d,g 179=
 =A,abcdg(3):a,b,d,e,g,h 180=168,defg:a,d,f,h 181=A,adfg,
 befg:a,c,d,f,h 182=180,abcfg(3):a,d,h 183=135,134:a,d,g
 184=A,ade:a,b,f 185=B,bdfgh:a,b,c,f 186=B,cefgh:a,b,f 187=
 =abc,def:a,g 188=184,adefgh:a,b,d,f 189=187,abcdgh:a,d,e,g
 190=184,bdfg:a,b,c,f,h 191=187,42:a,g,h 192=A,adgh:a,b,g
 193=184,bfgh:a,b,c,d,f 194=A,afgh:a,b,f 195=A,bdfg:a,b,c,f,h
 196=B,cefg:a,b,f,h 197=187,defgh:a,d,g 198=abcfg(3),184:a,b,
 d,f,h 199=A,168:a,d,h 200=187,abcdg(3):a,d,e,g,h 201=A,adg:
 a,b,g,h 202=134,184:a,b,f,g, 203=A,187:a,g 204=184,fg:a,
 b,f 205=A,afg:a,b,h 206=A,bdf:a,c,g 207=B,cef:a,g 208=
 =abcd(2):a,e 209=A,aefgh:a,b,e 210=A,42:a,e,h 211=A,aefg:
 :a,b,e,h 212=A,efgh:a,e 213=A,134:a,e,g 214=abc,defg:a,d,h
 215=A,adef:a,b,d,g 216=abcde(2):a,f 217=abc,adefgh:a,b,d

$(k, r) = (8, 6)$

1=U:a 2=abcdefg:a,h 3=abcdef:a,g 4=A,abcdgh:a,e 5=B,abefgh:
 :a,c 6=B,cdefgh:a 7=abcde:a,f 8=A,2:a,f,h 9=A,abcfgh:a,d,f
 10=B,adefgh:a,b,f 11=7,abcfg:a,d,h 12=A,abfgh:a,c 13=B,
 cdefgh:a,c 14=11,adefg:a,b,h 15=abcdef(4):a,g 16=abcd:a,e
 17=A,abef:a,c,g 18=A,aefg:a,b,h 19=A,efgh:a 20=17,cdef:a,g
 21=16,abefg:a,c,e,h 22=A,aefgh:a,b,e 23=21,cdefg:a,e,h 24=
 =16,15:a,e,g 25=A,abefgh:a,c,e 26=B,cdefgh:a,e 27=16,2:a,e,h
 28=abcde(3):a,f 29=abc:a,d 30=A,28:a,d,f 31=A,15:a,d,g 32=
 =A,2:a,d,h 33=A,adefgh:a,b,d 34=29,def:a,g 35=A,ade:a,b,f
 36=A,defg:a,d,h 37=A,adef:a,b,d,g 38=A,defgh:a,d 39=A,adefg:
 :a,b,d,h 40=abcd(2):a,e

$(k, r) = (8, 7)$

1=abc:a,d 2=abcd:a,e 3=abcde:a,f 4=abcdef:a,g 5=abcdefg:
:a,h 6=U:a

$(k, r) = (7, 3)$

1=abcdef:aa,gg,ab,ag 2=3,ceg:aa,ab 3=18,bdg,bef,cdf:aa,cc,
ab,ce,ac 4=9,afg:aa,bb,ff,ab,af,bc,fg,bf 5=19,afg,beg,cef:
:aa,dd,ee,ab,ef,ad,de,ag,ae 6=18,bdf,beg:aa,cc,dd,ab,de,dg,
ac,cd,ad 7=19,aefg:aa,bb,ab,bc,be 8=16,beg,dfg:aa,ee,ad,ae,
ab,af 9=abcde:aa,ff,ab,fg,af 10=14,dfg:aa,bb,ee,ff,ad,bc,
eg,ab,ae,af,ef,ag,be,bf 11=21,bef,cdf:aa,gg,ab,af,ag 12=16,
dfg:aa,bb,ee,ad,ac,af,ab,bg,ag,ae,be 13=18,cef:aa,bb,cc,ab,
ac,ce,bd,be,bc 14=19,aef:aa,bb,ee,gg,bc,ef,ab,ae,ag,bg,eg,be
15=A,efg:aa,ee,ab,ef,ae 16=21,cef:aa,bb,gg,ac,bd,ag,bg,af,ab
17=A,dfg:aa,bb,ee,ad,bc,bf,ae,be,ab,af 18=A,afg:aa,bb,ab,bc
bd 19=abcd:aa,ee,ab,ef,ae 20=22,def:aa,gg,ab,ag,ad 21=A,
ade:aa,bb,ff,bc,bd,af,bf,ab,fg 22=abc:aa,dd,ab,de,ad 23=U:
:aa,ab

$(k, r) = (7, 4)$

1=abcde(2):9.3 2=18,bef,cdf:11.3 3=10,aef:14.3 4=A,efg:15.3
5=18,cef:16.3 6=A,dfg:17.3 7=A,afg:18.3 8=10,aefg:aa,bb,ee,
bc,ef,ab,ae,be 9=A,33:aa,ee,gg,ab,ef,ag,eg,ae 10=abcd(2):
:19.3 11=13,33:20.3 12=A,adefg:aa,bb,dd,gg,bc,de,ag,ab,ad,
bg,dg,bd 13=31,def:20.3 14=18,33:aa,bb,ff,gg,bc,bd,af,ag,fg,
ab,bf,bg 15=17,bdfg:aa,bb,ff,bc,bd,be,fg,ab,af,bf 16=18,
abcfg:aa,bb,dd,ff,bc,de,fg,ab,ad,af,bd,bf,df 17=A,cefg:aa,bb,
cc,ff,bd,ce,fg,bc,be,ab,ac,af,bf,cf 18=31,ade:21.3 19=A,33:
aa,dd,gg,ab,de,ag,dg,ad 20=24,defg:aa,dd,ad,ab,de,df 21=25,
cefg:aa,bb,dd,ff,ac,de,fg,ab,bd,bf,df,af,ad,ae 22=27,cdfg:aa,
dd,ee,ad,de,ag,ae,ab,ef 23=31,adefg:aa,bb,dd,bc,de,bd,ab,ad
24=28,abcfg:aa,dd,ab,de,df,ad 25=A,adfg:aa,bb,dd,ee,ff,bc,fg,
ab,ad,ae,af,bd,be,bf,de,df,ef 26=A,defg:aa,dd,ff,ab,de,fg,ad,
df,af 27=29,bdeg:aa,cc,dd,ff,ab,de,fg,ac,cd,cf,ad,af,ag,df
28=31,abcde:aa,dd,ff,ab,de,fg,ad,af,df 29=A,adef:aa,bb,dd,gg,
ab,ad,ag,bd,bg,dg,bc,de 30=A,defg:aa,dd,ab,de,ad 31=abc:22.3
32=34,acfg:aa,ab 33=abcdef:1.3 34=36,adeg:aa,bb,ac,bd,ab

35=48,abef,cdefg:aa,cc,gg,ab,cd,ce,ac,ag,cg 36=38,bceg:aa,bb,
 cc,ag,cf,cd,ab,ac,bc 37=A,aceg:aa,gg,ag,ab,ac,ad 38=45,abef:
 :aa,bb,gg,ac,bd,ag,bg,ab,ad 39=43,ddefg:aa,dd,ff,ab,de,fg,ad,
 af,df 40=44,adfg:aa,bb,dd,ac,ag,ae,df,ab,bd,ad,af 41=45,
 adeg:aa,dd,af,ad,ab 42=47,abef:aa,gg,ab,ag,ac 43=abcde:9.3
 44=47,befg:aa,bb,cc,ag,cd,ce,ab,bc,ac,ae 45=A, bdfg:aa,bb,dd,
 ae,bc,ad,bd,af,ab 46=48,aefg:7.3 47=A,cdef:aa,cc,gg,ab,ae,
 cd,ag,cg,ac 48=abcd:19.3 49=U:23.3

(k,r) = (7,5)

1=abcd(2);19.3 2=9,def:20.3 3=9,ade:21.3 4=9,abcde(3):28.4
 5=9,adef:29.4 6=9,ddefg:30.4 7=9,abcdef:19.4 8=9,adefg:23.4
 9=abc:31.4 10=13,abef:42.4 11=abcde(3):9.3 12=17,aefg:46.4
 13=A,cdef:47.4 14=16,abefg:aa,ee,ab,ac,ef,ae 15=17,abcdef:
 :aa,ee,gg,ab,ef,ag,eg,ae 16=A,cdefg:aa,cc,ee,ab,cd,ef,ac,ae,
 ce 17=abcd:1.5 18=abcdef:33.4 19=20,adefg:aa,bb,ab,bc,bd
 20=21,abcfg:aa,dd,ab,de,df,ad 21=abcde:9.3 22=U:49.4

(k,r) = (7,6)

1=abc:22.3 2=abcd:19.3 3=abcde:9.3 4=abcdef:33.4 5=U:23.3

(k,r) = (6,3)

1=abcde:aaa,fff,aab,aaf,aff,abc,abf 2=abc,ade,bef,cdf:aaa,
 aab,aaf,abc,acd,abd 3=abcd,aef:aaa,bbb,eee,aab,aae,bba,eea,
 bbc,eeb,eef,bbe,abc,bcd,aef,bef,abe,bce 4=abc,ade,cef:aaa,
 bbb,aac,bbd,aab,bba,aaf,bbe,ace,bdf,abc,acd,abd,abf 5=abcd:
 :aaa,eee,aab,eef,aae,eee,abc,abe,aef 6=abc,def:aaa,aab,aad,
 abc,abd 7=abc,ade:aaa,bbb,fff,aab,bba,bbc,bbd,aaf,bbf,ffb,
 bcd,abc,abd,bcf,bdf,abf 8=abc:aaa,ddd,aad,dda,aab,dde,abc,
 def,abd,ade 9=U:aaa,aab,abc

(k,r) = (6,4)

1=abcd(2):5.3 2=abc,def:6.3 3=abc,ade:7.3 4=abc,abcde:aaa,
 ddd,fff,aab,dde,aaf,ffa,aad,dda,ddf,ffd,abc,abd,abf,ade,adf,
 def 5=abc,adef:aaa,bbb,ddd,aab,aad,bbc,abb,add,dde,bbd,bdd,
 def,abc,abd,bcd,bde,ade 6=abc:8.3 7=abcd,abef,cdef:aaa,aab,

aac,abc,ace 8=abcde:1.3. 9=abcd,abef:aaa,ccc,aab,ccd,cce,
aac,cca,abc,acd,cde,ace 10=abcd:5.3 11=U:9.3

(k,r) = (6,5)

1=abc:8.3 2=abcd:5.3 3=abcde:1.3 4=U:9.3

(k,r) = (5,3)

1=abcd:aaaa,eeee,aaab,aaae,eeea,aabb,aaee,aabc,aabe,eeab,abcd,
abce 2=abc,ade:aaaa,bbbb,aaab,bbba,bbbc,bbbd,aabb,bbcc,bbdd,
aabc,aabd,bbac,bbde,bbcd,bbad,abcd,bcde 3=abc:aaaa,ddd,aaab,
ddde,aaad,add,aaabc,aade,aabd,ddab,ddae,abcd,abde,aabb,ddee,
aadd 4=U:aaaa,aaab,aabb,aabc,abcd

(k,r) = (5,4)

1=abc:3.3 2=abcd:1.3 3=U:4.3

(k,r) = (4,3)

1=abc:aaaaa,ddddd,aaaab,aaaad,ddda,aaabb,aaadd,ddaa,aaabc,
aaabd,ddab,abbcc,abbdd,daabb,aabcd,abcd 2=U:aaaaa,aaab,
aaabb,aaabc,aabbc,aabcd

4. A LIST OF THE NUMBER OF SEMISIMPLE MATROIDS ON 9 ELEMENTS

Summing up the numbers of semisimple matroids with
(k,r) fixed, we obtain the following list:

r \ k	8	7	6	5	4	3	2	1	Σ
1								1	1
2		1	2	3	5	6	7	4	28
3	241	159	88	50	22	7			567
4	2751	393	110	33	6				3293
5	779	140	29	5					953
6	101	21	3						125
7	11	2							13
8	1								1
Σ	3885	717	233	93	34	14	4	1	4981

Thus, there are 4981 non-isomorphic semisimple matroids on 9 elements. In addition, according to the results of [5], there are 1724 non-isomorphic matroids with loops on 9 elements. It follows that the total number of non-isomorphic non-simple matroids on 9 elements is 6705.

5. "POSITIONAL" PARTITIONS

Non-isomorphic semisimple matroids on 9 elements which correspond to a given simple matroid M on k elements ($8 \geq k \geq 4$) induce a partition \sim of the set of adjoined unordered words of length $9-k$ (the words in the same class of \sim determine isomorphic matroids). Paper [5] contains a number of examples, which illustrate an effective construction of the classes of \sim , while an effort was made in [2] to reduce \sim to some simpler partitions (of course, the considerations in these two papers were not restricted to the semisimple matroids on 9 elements).

Generally speaking, classes of \sim seem to correspond to the orbits of automorphism groups defined on the sets of unordered words, over the alphabet consisting of atoms of simple matroids. The notion "positional partition", used for \sim in [2], reflects the fact that the classes of \sim depend on the relative positions of atoms and their combinations, within a simple matroid, with respect to the ranked essential flats.

6. CONSTRUCTION TECHNIQUES

We shall proceed by illustrating some techniques (at least we may call them "shortcuts"), which were used during our construction (=determination of classes of \sim). Their main role was to help our "intuitive" detection.

(1) Suitable matroid representations

The essential flats are easy to extract from the lattices of cyclic flats (the representations used in [4]); their existence cannot be "predicted" from below by using semimodularity. However, there exist some classes of paving matroids,

where essential flats are not easy to deal with. In such cases we use some more adjusted representations, like the Euclidean (rank 3 case), or special graphic representations, introduced in [3] (rank 4 case). It is helpful to visualize the orbits when possible.

(2) Frequencies

Given $k=8$, the frequency $f(x)$ of an atom $x \in \{a, b, \dots, h\}$ is the number of appearances of x in the essential flats of a simple matroid. It is also of interest to consider the restrictions of $f(x)$ to the essential flats of fixed cardinality and rank. It is obvious that atoms with different frequencies cannot be in the same class of \sim .

(3) Gradual "cutting up" of the classes

It often happens that by using one criterion, we obtain classes which should be cut up afterwards, because of some other criterion. For example, the frequencies give an "initial" partition $abcegh-df$ for the simple matroid

$397=abcd, abef, abgh, aceg, bceh, cdgh, efg,$

but the final partition is $abgh-ce-df$ (e.g. note that e and d , as well as c and f , never appear together in an essential flat).

The same "initial" partition can be used for many simple matroids. Thus the initial partition $abc-defgh$ applies to all the rank 4 simple matroids on 8 elements numbered from 157 to 199.

Observe that an augmentation (introducing some new essential flats) may cut, but also may join some previously existing classes. For example, the matroids 599 and 600 ($(k, r) = (8, 4)$) were obtained by two successive augmentations of the matroid 454. However, the matroids 454 and 600 have the same four classes of \sim , while the middle one (599) has eight classes.

(4) Duality and complementarity

LEMMA. If the dual M^* of a simple (not self-dual) matroid M is also simple, then the classes of \sim with respect to M and M^* coincide.

PROOF. Given a simple matroid M , the classes of \sim are shown in [2] to be equal to the orbits of the automorphism group $\text{Aut}(M)$. On the other hand, it is obvious that $\text{Aut}(M) = \text{Aut}(M^*)$ (if a permutation of the ground-set fixes the family of bases of M , then it necessarily fixes the family of their complements, i.e., bases of M^*). \square

This lemma gives, e.g. a control of the results obtained for 159, 65 and 20 pairs of mutually dual pairs for the (k,r) equal to $(8,4)$, $(8,3)-(8,5)$ and $(7,3)-(7,4)$, respectively. However, the matroids in [4] were not constructed by using dualization. Thus the orbits corresponding to mutually dual matroids in the catalogue need not be identical, but must be the same size.

A similar reasoning can be applied to the matroids numerated from 348 to 433 $((k,r)=(4,8))$. They are shown in [3] to be subfamilies of the Steiner system $S(3,4,8)$, the 14 blocks of which are the essential flats of the matroid 433. It is possible to use the complementary families (with respect to $S(3,4,8)$) when searching for the orbits in cases where there exist more than 7 essential flats.

(5) Number of combinations

The sum of orbit sizes must be equal to the number of combinations with repetitions of length $9-k$ over a k -element set, that is, $\binom{8}{9-k}$ (this equals 8, 28, 56, 70, 56, respectively, for $8 \geq k \geq 4$). This number may also serve as a control of the performed partitionings.

EXAMPLE. The cardinalities of classes of \sim , associated with the rank 3 matroid on the 4-element set abcd and the only essential flat abc are $(3,1), (6,3,3), (6,3,3), (3,6,3), (3,6,3)$, $(3,1)$, and their sum is 56. The brackets correspond to the words which have the same partitions of the numbers of different letters $(5+1, 4+2, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, respectively). The lexicographical ordering of the classes corresponding to the same partition is transferred from the lexicographically first unordered words in them.

(6) Strongly related elements

Given $k=8$, two atoms x and y are strongly related if each essential flat F satisfies: If $|F \cap \{x,y\}| = 1$, then there exists essential flat $(F \setminus \{x,y\}) \cup (\{x,y\} \setminus F)$. This "strong" relation equals the strong positional partition in [2]. It is shown in [2] that two strongly related elements must be in the same class of \sim (the converse is not true).

(7) Coloops

The coloops of a simple matroid necessarily constitute a special class of \sim .

(8) Equal partitions

As it has already been suggested by the denotation "n.r", recognizing (in advance) that two matroids have the same classes of \sim (for example, when their families of essential flats coincide, but the ranks do not) can save some work. However, it might be of particular interest to study the classes of equipartitional simple matroids (we suggest the matroids with only one class of \sim for the beginning).

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REZIME

JEDNA KONSTRUKCIJA SVIH NEIZOMORFNIH
NEPROSTIH MATROIDA NA 9 ELEMENATA

Konstruisano je (bez upotrebe kompjutera) svih 6705 neizomorfnih neprostih matroida na 9 elemenata. Konstrukcija je zasnovana na katalogu [4] svih neizomorfnih matroida na najviše 8 elemenata. Rad je nastavak rada [5] i korišćena je slična metoda, ali su oznake nešto skraćene. Rad sadrži katalog svih 4981 matroida bez petlji u posmatranoj klasi.

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