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ON AN ITERATIVE METHOD FOR SOLVING CERTAIN NONLINEAR SYSTEMS

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ABSTRACT

The purpose of this paper is to solve the system of nonlinear equations $Gx = 0$, where $G: \mathbb{R}^n \to \mathbb{R}^n$ is F-differentiable function and $G'(x)$ is a symmetric positive definite matrix for any $x \in \mathbb{R}^n$. These hypotheses imply that there exists a unique point $x^* \in \mathbb{R}^n$ such that $Gx^* = 0$. In order to solve this problem, we use the MSORN (Modified Successive Overrelaxation Newton) method, sufficient conditions for global convergence of which are given.

1. INTRODUCTION

We shall consider the system of nonlinear equations

$$Gx = 0, \tag{1}$$

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where G: $\mathbb{R} \to \mathbb{R}$ is F-differentiable function and G'(x) is a symmetric positive definite matrix for any $x \in \mathbb{R}^n$. These assumption imply that there exists a unique solution $x^* \in \mathbb{R}^n$, to equations $Gx^* = 0$.

A well known method for the solving our system is the nonlinear SOR method, which can be writen in the following form:

$$x_i^{k+1} = x_i^k + \omega(x_i - x_i^k), \quad i \in \mathbb{N}, \quad k=0,1,\dots,$$

where x_i is the solution of the equation

$$g_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \dots, x_n^k) = 0 \tag{2}$$

ω is a real parameter, $\omega \neq 0$ and g_i is the i-th component of G. In each step, the nonlinear equations (2) can be solved, for example, by using the Newton method. We decide to take just one step of Newton method, so that we obtain the SORN (Succesive Overrelaxation Newton) method:

$$x^0 \in \mathbb{R}^n, \ x_i^{k+1} = x_i^k - \omega p_i(x^{k,i}), \ i \in N := \{1,2,\dots,n\}, \ k=0,1,\dots,$$

where

$$x^{k,i} = [x_1^k, \dots, x_{i-1}^k, x_i^k, \dots, x_n^k]^T, \quad i \in \mathbb{N}$$

$$p_i(y) = \frac{g_i(y)}{\frac{\partial g_i}{\partial x_i}(y)}, \quad y \in \mathbb{R}^n .$$

Note that $\frac{\partial g_i}{\partial x_i}(x) > 0$ for all $i \in N$ and $x \in \mathbb{R}^n$. Instead of functions $\frac{\partial g_i}{\partial x_i}(x)$ we can use some positive continuous functions $d_i(x)$, so we obtain the Modified SORN method (MSORN), see [2]:

$$x^0 \in \mathbb{R}^n, \ x_i^{k+1} = x_i^k - \omega p_i^*(z^{k,i}), \ i \in N, \ k=0,1,\dots,$$

where

$$p_i^*(y) = \frac{g_i(y)}{d_i(y)},$$

and the other notations are as above.

2. MAIN RESULT

The system (1) with symmetric positive definite Jacobian matrix has a following proprty: there exists a strictly convex functional $F: \mathbb{R}^n \to \mathbb{R}$, so that $\frac{\partial F}{\partial x_i}(y) = g_i(y)$, $y \in \mathbb{R}^n$. Hence, our method can be applied to the problem of finding the minimum of strictly convex functional, but by using this method we can solve the system of type (1) without knowledge the functional F. In [1] was considered SORN method for finding minimum of strictly convex functional, and one can consider our MSORN method as a modification of this method. The proof of the following Theorem about global convergence of MSORN method is based on Theorem 1 from [1].

THEOREM. *Suppose that* $G : \mathbb{R}^n \to \mathbb{R}^n$ *is F-differentiable function and that* $G'(x)$ *is symmetric positive definite matrix for all* $x \in \mathbb{R}^n$. *Assume, further, that there are constants M,D and d such that*

$$\frac{\partial g_i}{\partial x_i}(x) \le M \ , \ 0 < d \le d_i(x) \le D \ , \ \forall x \in \mathbb{R}^n, \ \forall i \in N.$$

Then for any $\omega \in (0, \frac{2d}{M})$ *and* $x^0 \in \mathbb{R}^n$ *the iteration*

$$x_i^{k+1} = x_i^k - \omega p_i^*(x^{k,i}) \ , \ i \in N, \ k=0,1,\ldots, \tag{3}$$

$$x^{k,i} = [x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i^k, \ldots, x_n^k]^T,$$

converges to the solution of $Gx = 0$. ■

P r o o f: Let F be a strictly convex functional with property (see [3])

$$F_i(y) = \frac{\partial F}{\partial x_i}(y) = g_i(y),\ y\in \mathbb{R}^n.$$

Let $F_{ii}(y) = \frac{\partial^2 F}{\partial x_i^2}(y)$, then we have

$$p_i^*(y) = \frac{g_i(y)}{d_i(y)} = \frac{F_{ii}(y)}{d_i(y)}\ .$$

Let $z^{k,i}$ be a point such that

$$F(x^{k,i}) = F(z^{k,i})\ ,\quad x_j^{k,i} = z_j^{k,i}\ ,\ j \neq i,\ \text{ and } x_i^{k,i} \neq z_i^{k,i} \ \text{ unless } F_i(x^{k,i}) = 0.$$

From Taylor's theorem we have, for some $\alpha^{k,i}$ between $x^{k,i}$ and $z^{k,i}$,

$$F(z^{k,i}) = F(x^{k,i}) + (z_i^{k,i} - x_i^{k,i})F_i(x^{k,i}) + \frac{1}{2}(z^{k,i} - x_i^{k,i})^2\, F_{ii}(\alpha^{k,i})\ ,$$

and we conclude that

$$z_i^{k,i} - x_i^k = z_i^{k,i} - x_i^{k,i} = \frac{-2F_i(x^{k,i})}{F_{ii}(\alpha^{k,i})}.$$

Now, the MSORN method can be written as

$$x^0\in\mathbb{R}^n,\quad x_i^{k+1} = x_i^k + r^{k,i}(z_i^{k,i} - x_i^k)\ ,$$

where

$$r^{k,i} = \frac{\omega F_{ii}(\alpha\quad)}{2d_i(x^{k,i})}\ .$$

Because of Theorem 1 from [1], for global convergence of MSORN method it is sufficient to show that there exists $\tau\in(0,\frac{1}{2}]$, such that $\tau \le r^{k,i} \le 1 - \tau$, for all k and i. Obviously, $0 < r^{k,i} < 1$, so that $x^k\in S:=\{x\in\mathbb{R}^n \mid F(x) \le F(x^0)\}$, for all $k=0,1,2,\ \dots$. Since S is compact set, there exists a positive constant m, for which is $F(x) \geq m$, for all x and i. Let $0 < \omega \leq \frac{2Dd}{md + MD}$. If we put $\tau = \frac{\omega m}{2D}$, we can easy verify that $\tau \in (0, \frac{1}{2})$ and $\tau < r^{k,i} \leq 1 - \tau$, for all k and i. If $\frac{2Dd}{md + MD} \leq \omega < \frac{2d}{M}$, we should choose $\tau = 1 - \frac{\omega m}{2d}$ in order to obtain $\tau \in (0, \frac{1}{2}]$ and $\tau \leq r^{k,i} \leq 1 - \tau$, for all k and i. So, we conclude that MSORN method converges for any $x \in \mathbf{R}$, if we choose $\omega \in (0, \frac{2d}{M}]$. □

3. NUMERICAL RESULTS

In this section, we present some numerical results in two dimensional case. We deal with the same problem as in [1].

We wish to solve the system of equations

$$g_1(x_1,x_2) := \mathrm{arctg}(x_1 + x_2) = 0,$$

$$g_2(x_1,x_2) := \mathrm{arctg}(x_1 + x_2) + 2x_2 = 0,$$

the solution of which is obviously $[0,0]^T$. It is easy to see that

$$\frac{\partial g_1}{\partial x_1}(x) = \frac{1}{1 + (x_1 + x_2)^2},$$

$$\frac{\partial g_2}{\partial x_2}(x) = \frac{1}{1 + (x_1 + x_2)^2} + 2 .$$

Let $d_1(x) = 1$ and $d_2(x) = 2$. Then with $M = 3$, $d = 1$ and $D = 2$ assumptions of our Theorem are satisfied and MSORN method converges for $\omega \in (0,2/3)$.

Let $X = \{x \in \mathbf{R}^2: \max(|x_1|,|x_2|) \leq 10\}$, $X_h = \{(k/2, j/2): k,j = -20, -19, -18,\ldots,-1,0, 1,2,\ldots, 18,19,20\}$, $\Omega = \{k/8: k=1,2,\ldots,15\}$. The numerical experiments show that MSORN method convergens for any starting vector from X_h if $\omega = k/8$, $k=1,2,\ldots, 13$. The convergence area of SORN method for some values $\omega \in \Omega$ is shown in Figure 1. But, for $x \in X$ we have $m = \frac{1}{401}$, and from Theorem we can conclude that SORN method converges for $\omega \in (0, \frac{2}{3} \frac{1}{401})$. One can observe that the convergence area of MSORN method is greater than the convergence area of SORN method for $\omega \in \Omega \setminus \{14/8, 15/8\}$.

For each $\omega \in \Omega$ we have applied SORN and MSORN method with each vector from X_h as the starting vector. In this case, for fixed ω we have 1681 different starting vectors. For each starting vector x^0 is obtained the number of iterations $k = k(\omega, x^0)$ necessary to achieve the numerical convergence criterion

$$\|x^k\|_\infty < 2^{-10} .$$

For each $\omega \in \Omega$ there is $k_\omega = \min\{k(\omega, x^0) : x^0 \in X_h\}$. Table 1 gives the numbers k_ω and corresponding starting vectors for both methods SORN and MSORN.

Table 1.

	SORN			MSORN		
ω	x_1^0	x_2^0	k_ω	x_1^0	x_2^0	k_ω
1/8	-5.5	0.0	65	-0.5	-0.5	24
2/8	-3.0	0.0	36	-0.5	-0.5	21
3/8	-3.5	0.5	10	-2.0	-4.0	15
4/8	-3.0	0.5	14	-1.5	-2.5	10
5/8	-1.0	1.0	8	-1.5	-3.5	9
6/8	-3.5	1.5	11	-1.5	-3.5	5
7/8	-0.5	0.0	4	-1.0	-4.0	4
8/8	-2.5	1.5	3	-0.5	0.0	3
9/8	-0.5	0.0	4	-0.5	0.0	4
10/8	-0.5	0.0	5	-1.0	0.0	6
11/8	-0.5	0.0	6	-1.0	0.0	12
12/8	-0.5	0.0	9	-1.5	0.0	15
13/8	-0.5	0.0	14	-1.5	0.0	85
14/8	-0.5	0.0	22			
15/8	-0.5	0.0	66			

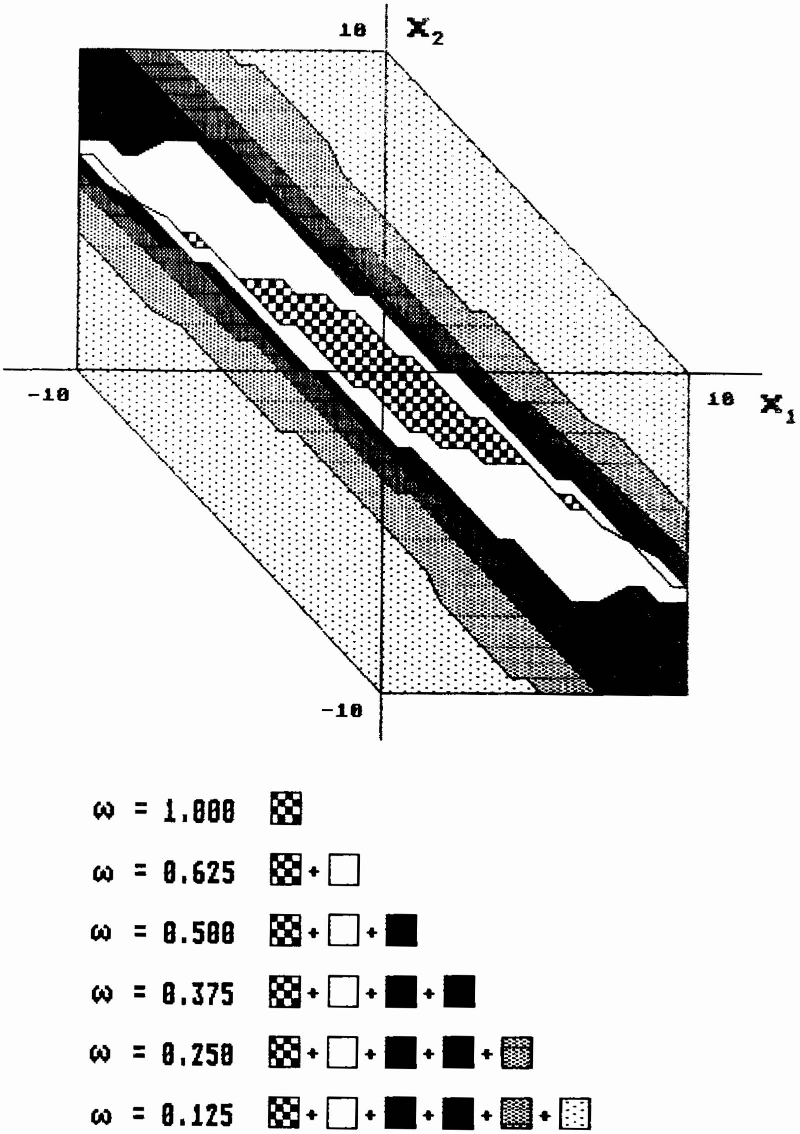

Figure 1.

We can note that numbers k_ω for MSORN and SORN method are similar for $\omega \in [1/8, 10/8]$. But, for small ω, the numbers k_ω for MSORN method are smaller than the corresponding numbers of SORN method.

4. CONCLUSIONS AND REMARKS

The MSORN method can be used for finding the minimum of strictly convex functional. It seems to us that computational procedure for this method is much simpler than those one from [1]. However, how long we succesed to compare numerical results for both methods, the convergence of the MSORN method was better.

MSORN method has three adventages connected the SORN method: first, if ω is choosen so that the global convergence is garanteed, MSORN method might converge faster; second, if ω does belong to the interval of the global convergence, the permitted convergence area for a starting vector can be significantly greater in case of the MSORN method and third, the MSORN method can be used without knowelidge the diagonal elements of the Jacobian matrix.

The assumptions which are related to function G can be weaker from $\mathbb{R}^n$ to $S = \{x \in \mathbb{R}^n | F(x) \le F(x^0)\}$ if we know the functional F.

The MSORN method can be applied in order to solve nonlinear systems, for which we don't know the functional F.

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REZIME

O JEDNOM ITERATIVNOM POSTUPKU ZA REŠAVANJE ODREĐENIH NELINEARNIH SISTEMA

Rešava se sistem nelinearnih jednačina $Gx = 0$, gde je $G:\mathbb{R}^n \to \mathbb{R}^n$ F-diferencijabilno preslikavanje, a $G'(x)$ simetrična pozitivno definitna matrica za svako $x \in \mathbb{R}^n$. Koristi se MSORN (Modifikovani SOR Njutnov postupak) i daju se dovoljni uslovi za njegovu globalnu konvergenciju.

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