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ON THE METHOD OF AVERAGING FUNCTIONAL CORRECTIONS APPLIED TO LINEAR SYSTEMS

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ABSTRACT

The method of averaging functional corrections (AFC) applied to the solution of linear systems is considered. The matrix form of this method is obtained, so that new sufficient conditions for the convergence of AFC method, which are weaker than the known one, are proved.

1. INTRODUCTION

The method of averaging functional corrections (AFC) for solving a system of linear equations

$$x = Ax + b, \quad A = [a_{ij}] \in \mathbb{R}^{n,n}, \quad b = [b_i] \in \mathbb{R}^n \tag{1}$$

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is of the form

$$x^0 \in \mathbb{R}^n, \quad x^{k+1} = A(x^k + y^k) + b, \quad k = 0,1,\dots, \qquad \text{(AFC)}$$

where

$$y^k = s_k[1,1,\dots,1]^T \in \mathbb{R}^n, \quad s_k = \frac{1}{n} \sum_{i=1}^{n} (x^{k+1} - x^k) .$$

A computational procedure for computing the approximations x^k of the solution of system (1) (given in [4]) is:

Step 0: Calculate $a = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$;

Step 1: If $n \le a$ stop, otherwise go to step 2;

Step 2: Choose $x^0 \in \mathbb{R}^n$;

Step 3: Calculate $s_0 = \frac{1}{n-a} \sum_{i=1}^{n} b_i$; $k = 0$;

Step 4: Calculate x^{k+1} by AFC method (3);

Step 5: Calculate $s_{k+1} = \frac{1}{n-a} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_j^{k+1} - x_j^k - s_k)$;

Step 6: Take $k = k+1$ and return to step 4.

2. THE MATRIX FORM OF THE AFC METHOD

We allways suppose that $|a| < n$ and we shall see this assumption is not restrictive.

Let us denote by $P \in \mathbb{R}^{n,n}$ the matrix, all entries of which are equal to 1. Then the AFC method can be written in the following form:

$$x^{k+1} = A\left(x^k + \frac{1}{n} P(x^{k+1} - x^k)\right) + b, \quad k = 0,1,\dots,$$

i.e.

$$(E - \frac{1}{n}AP)x^{k+1} = (A - \frac{1}{n} AP)x^{k} + b \ , \ k = 0,1,\dots \ .$$

The matrix AP is of the form

$$AP = \begin{bmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \dots & a_n \end{bmatrix},$$

where

$$a_i = \sum_{j=1}^{n} a_{ij} \ , \quad i = 1,2, \dots , n.$$

It is easy to show that $\rho(AP) = |a|$. Hence, $\rho(\frac{1}{n}AP) < 1$, the matrix $E - \frac{1}{n}AP$ is nonsingular and

$$(E - \frac{1}{n}AP)^{-1} = \sum_{k=0}^{\infty} (\frac{1}{n}AP)^k.$$

Obviously,

$$\left(\frac{1}{n}AP\right)^2 = \frac{a}{n}\,\frac{1}{n}\,AP \ ,$$

and

$$(E - \frac{1}{n}AP)^{-1} = E + \frac{1}{n}AP\sum_{k=0}^{\infty} \left(\frac{a}{n}\right)^k = E + \frac{1}{n-a} AP \ .$$

Now, the AFC method has the following form:

$$x^{k+1} = Bx^{k} + \left(E + \frac{1}{n-a}AP\right)b \ , \ k=0,1,\dots \ .$$

where

$$B = \left(E + \frac{1}{n-a}AP\right)\left(A - \frac{1}{n}AP\right) .$$

After some calculations we conclude that the entries $(B)_{ij}$ of the matrix B are:

$$(B)_{ij} = a_{ij} - \frac{1}{n-a}\, a_i(1 - a_j^{*}) \ , \ i,j = 1,2,\dots, n,$$

where

$$a_i^* = \sum_{j=1}^{n} a_{ji}, \quad i = 1,2, \dots , n.$$

3. THE CONVERGENCE RESULTS

The following theorem from [2] and [4] containes the sufficient conditions for convergence of AFC method.

THEOREM 1. *Let* $h > \|A\|_\infty$, $z = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|$, $q = h + (1+h)\frac{z}{d}$

$$\omega = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(a_{ij} - \frac{1}{n}\sum_{k=1}^{n} a_{ik}\Big)^2 \left(1 + \frac{1}{n-a}\sqrt{n\sum_{i=1}^{n}\Big(\sum_{k=1}^{n} a_{ik}\Big)^2 - a^2}\right)^2.$$

Then AFC converges for any $x^0 \in \mathbb{R}^n$ *if* $q < 1$ *or* $\omega < 1$, *and* $n > a$. ■

Note that the condition $|a| < n$ is a consequence of the assumptions of Theorem 1.

By using our matrix presentation of the AFC method, we are available to prove new sufficient conditions for the convergence of AFC method, which are weaker then those two from above theorem. For example, it is easy to see that the following Theorem is true.

THEOREM 2. *If* $|a| < n$ *and* $\|B\|_\infty < 1$ *or* $\|B\|_1 < 1$, *then AFC converges for any* $x^0 \in \mathbb{R}^n$. ■

Since the matrix B has constant row sums equal to 0, because for each i=1,2, ... , n it holds that

$$\sum_{j=1}^{n}(B)_{ij} = \sum_{j=1}^{n} a_{ij} - \frac{1}{n-a}\, a_i \sum_{j=1}^{n}(1 - a_j^*) = a_i - a_i = 0,$$

we can use the Zenger's upper bound for the spectral radius of the matrix B, [6], [3], and prove the following Theorem.

THEOREM 3. *Let* $|a| < n$ *and*

$$\delta = \frac{1}{2} \max_{i,j} \sum_{k=1}^{n} \left|(B)_{ik} - (B)_{jk}\right| < 1.$$

Then AFC converges for any $x^0 \in \mathbb{R}^n$. ■

P r o o f. Since the matrix B is a row-stochatic matrix, then for all its eigenvalues λ, different from 0, holds $|\lambda| \le \delta$. Hence $\rho(B) \le \delta < 1$. □

4. NUMERICAL EXAMPLE

The advantage of AFC method is very good seen in [2], where the example for each AFC method converges, while the basic method does not, is given.

Here, we shall illustrate how weaker are our convergent conditions (Theorems 2 and 3), then those ones from Theorem 1.

Let

$$A = \begin{bmatrix} 0.70 & 0.02 & 0.12 & 0.14 \\ 0.02 & 0.04 & 0.04 & 0.06 \\ 0.12 & 0.04 & 0.60 & 0.08 \\ 0.14 & 0.06 & 0.08 & 0.30 \end{bmatrix}.$$

Obviously, the condition $|a| < n$ is satisfied. We compute $q = 4.5$ and $\omega = 1.8354$, as well as $\|B\|_\infty = 1.3950$, $\|B\|_1 = 1.3333$, $\delta = 0.8364$. Note that $\rho(B) = 0.629942$ and $\rho(A) = 0.8327$.

REFERENCES

[1] Deutsch,E., Zenger,Ch., *Inclusion domains for the eigenvalues of stochastic matrices, Numer. Math. 18(1971), 182-192.*

[2] Luchka,A.Yu., *The Method of Averaging Functional Corrections: Theory and Applications, Academic Press, New York and London, 1965.*

[3] Seneta, E., *Non-negative Matrices and Markov Chains, Springer-Verlag, New York Heidelberg Berlin, 1981.*

[4] Сиренко,В.Х., *О численной реализации метода осреднения функциональных поправок, УМЖ 13(1961), 51-66.*

[5] Соколов,Ю.Д., *О методе осреднения функциональных поправок, УМЖ 9(1957), 82-100.*

[6] Zenger,Ch., *A comparation of some bounds for the nontrivial eigenvalues of stochastic matrices, Numer. Math. 19(1972), 209-211.*

REZIME

O POSTUPKU SREDNJIH FUNKCIONALNIH KOREKCIJA PRIMENJENOM NA LINEARNE SISTEME

U radu se posmatra numeričko rešavanje sistema linearnih jednačina postupkom srednjih funkcionalnih korekcija. Za taj postupak je dobijena matrična reprezentacija na osnovu koje su dati dovoljni uslovi konvergencije. Ovi uslovi su manje restriktivni od poznatih, što je ilustrovano i odgovarajućim numeričkim primerom.

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